HOUSEHOLD PRODUCTION AND WELFARE EVALUATION WITH

NON-CONSTANT RETURNS TO SCALE.

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I. INTRODUCTION

A recent body of literature examining energy-consuming durables investment [Hausman (1979), Cameron (1982), and Hartman (1984)] has fused the household production and utility maximization models to estimate demand functions for end-use products such as heated water and air. All of these studies have made the seemingly necessary simplifying assumption that the household production functions exhibit constant returns to scale. This assumption provides for a linear budget constraint, a necessary condition for deriving demand functions.

However, since it is unlikely that all the household production functions in fact closely approximate linearity, serious measurement error may occur. It is the purpose of this paper to prove that the constant returns to scale assumption is unnecessary. A marginal benefit function (or Marshallian demand function) for end-use products can be derived and estimated under a non-linear budget constraint. In addition, it will be shown that measures of consumer's surplus, both Marshallian and Hicksian, can also be estimated under non-linear budget constraints.

In sections II and III, we formally derive the Marshallian demand function and explain how it and consumer's surplus can be estimated. In section IV, Shephard`s lemma and Roy's identity are modified to account for the non-linear budget constraint and a method of estimating Hicksian (or exact) consumer's surplus is shown. Section V is an illustrative example of the described process and Section VI concludes the paper.

II. MARSHALLIAN DEMAND UNDER A NON-LINEAR BUDGET CONSTRAINT.

The first step is to define a utility function,

$$U = U(Z,G), \tag{1}$$

where Z denotes the quantity of the end-use product produced and consumed by the household. Z is assumed to be a normal good. G is a composite good including all other goods and services purchased or produced by the household. U is assumed to be weakly separable, monotonically increasing and strictly concave in Z and G.

Next, we define a budget constraint

$$Y = C(Z,P_1,\ldots,P_n)+G, \tag{2}$$

where Y is household income, C is the cost function of producing Z, and P_i, for i=1...n, is the price of the i $\underline{th}$ input. The price of G is assumed to be equal to one. C is assumed to be monotonically increasing in Z and the prices of the inputs.

Using (1) and (2), we set up our Lagrangian equation,

$$L = U(Z,G)+\lambda(Y-C(Z,P_1,...P_n)-G) \qquad (3)$$

This gives us the first-order conditions,

$$L_z = U_z - \lambda C_z = 0 \qquad (4)$$

and

$$L_G = U_G - \lambda = 0, \qquad (5)$$

where U_z and U_G are the marginal utilities of Z and U respectively. C_z is the marginal cost of Z.

From our first-order conditions we derive the maximization condition

$$\frac{U_z}{U_G} = C_z \qquad (6)$$

This condition states that at maximum utility, the marginal rate of substitution is equal to the marginal cost of Z. Since the price of G is set to one, C_z is also the ratio of the respective marginal costs of G and Z. Therefore (6) also states that at maximum utility, the marginal rate of substitution is equal to the marginal rate of transformation.

If constant returns to scale had been assumed, then C_Z would be constant in Z. The Marshallian demand function for Z would then be derived by substituting (2) into (6) and solving for Z. However, we allow for C_Z to be non-constant in Z, therefore substituting (2) into (6) and solving for Z results in

$$Z = F(P_1, \ldots, P_n, Y). \tag{7}$$

Since C_Z is not assumed constant in Z, F cannot properly be called a demand function, therefore it will be referred to as an uncompensated consumption function. However like a Marshallian demand function, F is assumed to be decreasing in price(s) and increasing in income.

That F is not a proper demand function would lead one to the correct conclusion that it cannot be thought of as a marginal benefit function. Thus consumer's surplus cannot be measured as the integral of F bounded by the ex-post and ex-ante prices. To determine consumer's surplus, a true marginal benefit function must be determined.

Since F plots out the combination of Z, Y and input prices that maximize utility, we know that at those points the marginal benefit must equal the marginal cost. To derive the marginal benefit function, we first invert F to solve for a price P_1.

$$P_1 = F^{-1}(Z, P_2, \ldots, P_n, Y), \tag{8}$$

and substitute (8) into C_z, the marginal cost,

$$B_z = C_z(Z, F^{-1}(Z, P_2, \ldots, P_n, Y), P_2, \ldots, P_n). \tag{9}$$

B_z is the uncompensated marginal benefit function which is analogous to the classical Marshallian demand function. The choice of P_1 is arbitrary since any input price will suffice. The chosen price will be called the "indirect price". In application, the indirect price would be the input price that changed, causing the level of consumption to change.

Graphically, this process is illustrated by figure (1). By varying P_1, the intersection of C_z and F "traces-out" B_z. As figure (1) illustrates, we have implicitly assumed a stable equilibrium, that is, the marginal cost curve has a greater slope than the marginal benefit curve.

Even though in this case there is no directly observable Marshallian demand function, the uncompensated consumption function and cost function, both of which are observable, can be used to determine a Marshallian marginal benefit function.

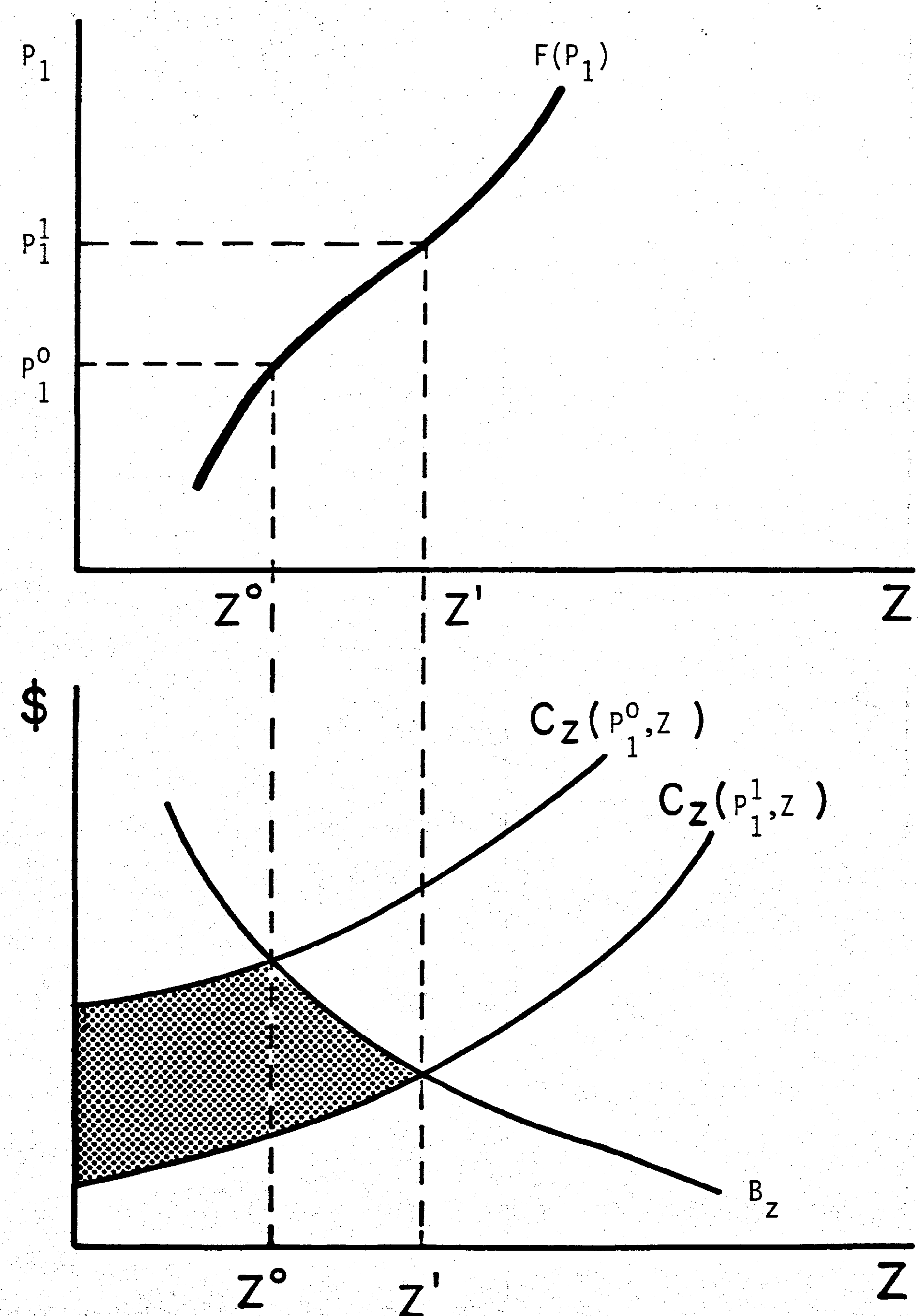

Figure (1)

III. MARSHALLIAN CONSUMER'S SURPLUS UNDER A NON-LINEAR BUDGET CONSTRAINT.

Now that we have derived a marginal benefit function, together with the marginal cost function, we can determine the change in consumer's surplus resulting from a change in an input price. A common example of this is the reduction of the cost of ceiling insulation and hence air-conditioning through rebate programs.

The change in consumer's surplus (dCS) is illustrated in figure (1) as the shaded area between the ex-ante and ex-post marginal cost curves and the marginal benefit curve. Mathematically this is

$$dCS = \int_{0}^{Z^0} (C_z(P_1^0,Z) - C_z(P_1^1,Z))dz + \int_{Z^0}^{Z^0} (B_z(z) - C_z(P_1^1,Z))dZ, \quad (10)$$

where Z^0 and P_1^0 denote ex-ante values and Z^1 and P_1^1 denote ex-post values. The other price arguments have been suppressed for notational simplicity.

(10) simplifies to

$$dCS = C(P_1^0,Z^0) - C(P_1^1,Z^1) + B(Z^1) - B(Z^0). \quad (11)$$

The first two terms in (11) sum to be the change in total cost. The last two terms in (11) sum to be the area under B_z bounded by Z^0 and Z^1. Thus, if the marginal benefit function is known the value of marginal cost at very low levels of production need not be known. Only the change in total cost, Z^0 and Z^1 need be observed.

IV. ROY'S IDENTITY AND SHEPHARD'S LEMMA UNDER A NON-LINEAR BUDGET CONSTRAINT

The change in the value of the expenditure function as a result of a change in price is known as exact consumer's surplus, a more refined welfare measure than Marshallian consumer's surplus. Hausman (1981) showed that Roy's identity could be utilized to determine the value of the expenditure function from the observed Marshallian demand function.

As we mentioned previously, the authors of applied papers in household production deemed it necessary to assume a constant marginal cost (linear-budget constraint). The need for this arose since they wished to use Roy's identity to determine a demand function for the end-use products. According to Pollak (1978) and more recently Bockstael and McConnell (1983), a linear budget constraint is necessary when utilizing Roy's identity and Shephard's lemma in the household production case.

Although it is correct that a constant shadow-price for output is necessary to apply Roy's identity and Shephard's lemma in their regular forms, it is not correct that analogous theorems for the non-linear case can not be proven.

To show that there is a way of estimating expenditure and indirect-utility functions and therefore an analogous Roy's identity and Shephard's lemma, in the non-linear budget constraint case, we first must redefine certain terms.

In the exogenous commodity price case, the indirect utility function is derived by maximizing direct utility subject to a linear budget constraint. Thus, the indirect utility function has as its arguments, the price of the goods consumed and income. This same process for the non-linear case results in the indirect utility function being a function of the input prices and income, which notationally is $v(P_1,\ldots,P_n,Y)$. To determine the expenditure function (e), we invert v to solve for Y. Thus e is a function of input prices and v, which notationally is $e(P_1,\ldots,P_n,v)$. Shephard's lemma tells us that if we know the expenditure function, we can determine a Hicksian (income-compensated) demand function. It is also a necessary result in order to prove Roy's identity. In the non-linear case, the objective is not to show the relationship between the expenditure function and income-compensated demand but the relationship between the expenditure function and the income-compensated consumption function (denoted by f). Therefore, the modified Shephard's lemma is

$$z = f(P_1,\ldots,P_n,v) = C_{p_1}^{-1}(P_1,\ldots,P_n,e_{p_1}(P_1,\ldots,P_n,v)). \quad (12)$$

$C_{P_1}^{-1}$ is the inverted partial derivative of the cost function with respect to P_1 solved for Z. e_{p_1} is the partial derivative of e with respect to P_1.

Proof:

Let us define a function,

$$x(P_1) = e(P_1,v) - C(P_1,Z^*) - G^*, \tag{13}$$

where Z^* and G^* are the expenditure minimizing amounts of Z and G that produce v level of utility when $P_1 = P_1^*$. For notational simplicity all other price arguments have been suppressed. Since e is the minimum expenditure necessary to get v, $x(P_1) \leq 0$. At $P_1 = P_1^*$, $x(P_1^*) = 0$. Since this is a maximum value,

$$X_{p_1}(P_1^*) = e_{p_1}(P_1^*,v) - C_{p_1}(P_1^*,Z^*) = 0. \tag{14}$$

From (14) we see,

$$e_{p_1}(P_1^*,v) = C_{p_1}(P_1^*,Z^*). \tag{15}$$

(15) states that for any given level of v, $e_{p_1} = C_{p_1}$ when the expenditure minimizing amount of Z is produced and consumed.

To derive an income-compensated consumption function (along which v is held constant), we need only solve (15) for Z,

$$Z = f(P_1,v) = C_{P_1}^{-1}(P_1,e_{P_1}(P_1,v)). \quad (16)$$

Q.E.D.

Now that we have a modified Shephard's lemma, we can establish a modified Roy's identity,

$$Z = F(P_1,Y) = C_{P_1}^{-1}\left(P_1, \frac{-V_{P_1}}{V_y}\right), \quad (17)$$

where V_{P_1} and V_Y are the partial derivatives of the indirect utility function with respect to P_1 and Y respectively.

Proof:

Let us define a function D to be the residual consumption function for the composite good G. Therefore, at maximum utility for the budget constraint $Y^* = C(P_1,Z) + G$, $Z^* = F(P_1^*,Y^*)$ and $G^* = D(P_1^*,Y^*)$. Let $U^* = U(Z^*,G^*) = U(F(P_1^*,Y^*), D(P_1^*,Y^*)) = v(P_1^*,Y^*)$. Since e is the inverse function of v, then $U^* = v(P_1,e(P_1,U^*))$. Since U^* is the maximal utility, we can differentiate it to get

$$V_{p_1}(P_1^*, Y^*) + V_y(P_1^*, Y^*) e_{p_1}(P_1, U^*) = 0 \quad (18)$$

or

$$e_{p_1}(P_1, U^*) = \frac{-V_{p_1}}{V_y}. \quad (19)$$

From our modified Shephard's lemma (equation (15)), we know that

$$e_{p_1}(P_1, U^*) = C_{p_1}(P_1, f(P_1, U^*)) \quad (20)$$

Since at maximum utility

$$f(P_1, U^*) = F(P_1, Y), \quad (21)$$

we see that

$$C_{p_1}(P_1, F(P_1, Y)) = \frac{-V_{p_1}}{V_y}, \quad (22)$$

or

$$Z = F(P_1, Y) = C_{p_1}\left(P_1, \frac{-V_{p_1}}{V_y}\right). \quad (23)$$

Q.E.D.

As Hausman has demonstrated, through Roy's identity the observation of the Marshallian demand function determines the parameter values of v and hence e also. In the non-constant returns to scale, household production case, the observation of the consumption function and the cost function together can determine the parameter values of v and e.

Although a more thorough analysis, utilizing income-compensated marginal benefit functions to determine the approximation error of Marshallian consumer's surplus versus exact consumer's surplus is possible, it is not included here. For such an analysis, the interested reader is referred to Scoggins (1985).

IV. ILLUSTRATIVE EXAMPLE

For our example we choose a Cobb-Douglas cost function

$$C(Z,P_1,\ldots,P_n) = AZ^{\alpha}\prod_{i=1}^{n} P_i^{\beta_i},$$

where $A>0$, $\alpha>0$ and $\beta_i>0$. The indirect utility function we choose was introduced in Hausman (1981).

$$v = Y^{\gamma} - B\prod_{i=1}^{n} P_i^{\delta_i},$$

where $B>0$, $\gamma>0$ and $\delta_i>0$. The resulting expenditure function is

$$e = (v + B \prod_{i=1}^{n} P_i^{\delta_i})^{1/\gamma}.$$

If we choose P_1 as the indirect price, then by applying the procedure detailed in section II, we can derive the uncompensated consumption function

$$Z = \left[\left(\frac{B^{\delta_i}}{A\gamma^{\beta_i}}\right)^{1/\alpha}\right] \left[\gamma^{(1-\gamma)/\alpha}\right] \left[\prod_{i=2}^{n} P_i^{\left(\frac{\delta_i-\beta_i}{\alpha}\right)}\right].$$

There are two main points to be made here. (1) The above equation is log-linear which simplifies the estimation of the parameter values. (2) The estimated parameters of the uncompensated consumption function and the cost function give us estimates of the expenditure function parameters. We, therefore, can estimate the change in exact consumer's surplus from a change in an input price, even when the returns to scale are not constant.

VII. CONCLUSIONS

The household production model has been shown to be very useful in the analysis of energy-consuming durables demand and the effectiveness of government sponsored energy conservation programs that alter the prices of these durables. If one thinks of each household as a separate producer of end-use services, such as cooled air, then one realizes the very small scale on which each household operates. The likelihood that the production technology might not exhibit constant returns to scale is high. Therefore, a new utility maximization paradigm must be invented to account for the non-linear budget constraint.

The result of this new paradigm is an observable consumption function which is not to be confused with an observable demand function. If the cost function is also observable, then it is possible to extract from the consumption function a marginal benefit function. This marginal benefit function serves the same purpose as a Marshallian demand function.

If a more precise (income-compensated) measure of marginal benefit is desired, then that too can be estimated. By modifying Shephard's lemma and Roy's identity for the non-linear case, a compensated consumption function can be derived. Roy's identity shows the relationship between the indirect utility function and the observable uncompensated consumption function. Knowledge of the parameters of the cost and uncompensated consumption functions is equivalent to knowledge of the parameters of the indirect utility and expenditure functions. Exact consumer's surplus can thus be estimated from observable phenomena.

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