COST ALLOCATION IN PUBLIC ENTERPRISES:
THE CORE AND ISSUES OF CROSS-SUBSIDIZATION

By

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Abstract

This paper examines the implications of core allocations on the provision of a service to a community and the corresponding cost allocation schemes and industrial structures. In the process it is shown that the existence of such allocations is neither a necessary nor a sufficient condition for the existence of subsidy-free cost allocation schemes. In addition, we propose a set of conditions that will make any notion of cross-subsidization "precise" and we introduce such notions.

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1. Introduction

The provision of a service to a given community raises a number of issues including (a) how the members of the community will share the relevant costs, and (b) how the provision of the service will be organized. Large overhead costs, common costs, joint costs, and long-run average costs that decrease over a relatively wide range of output, are some of the reasons often cited as part of the problem. This has prompted a number of authors to use a variety of solution concepts to n-person cooperative games, such as the Shapley value, the core, and the nucleolus, to study some of those issues.[1] While a game-theoretic approach can provide useful insights in addressing certain problems associated with the provision of a service, it would seem that some solution concepts do not necessarily have all those attributes reported in the literature.

A case in point is the core. In an appropriately defined cooperative game, the core can reveal what arrangements (output of the service, cost allocation schemes, and industrial structures), if any, can be considered stable, in a specific sense, for a given community with certain characteristics. But, does the core reveal whether the corresponding cost allocation schemes are subsidy-free as, for example, Faulhaber (1975), Littlechild (1975), and Sharkey (1982a,b) have claimed? Suppose, for example, that a community consists of members who are idential to each other in all respects, the same quantity of the service is provided to each member under identical cost arrangements, and the resulting allocation is Pareto optimal for the community. To conclude that the corresponding cost allocation scheme is

[1]See, e.g., the relevant sections in Schotter and Schwodiauer (1980), Sandler and Tschirhart (1980), Sharkey (1982b), and the various references therein.

subsidy-free only if this is a core-allocation, but not subsidy-free otherwise, it will be equivalent to confusing stability (in this case, core-stability) with cross-subsidization. No matter how vague a notion of cross-subsidization is, one should expect that in a situation like the above there is not cross-subsidization.

That the core of an appropriately defined cooperative game may not be sufficient to determine whether, in the provision of a service, cost allocation schemes are subsidy-free, raises two important questions: (a) what are the implications of core allocations on (i) the provision of a service, (ii) cost allocation schemes, and (iii) industrial structures? (b) what is cross-subsidization?

To address these questions the next section introduces a model that: (a) takes into account the income and preferences of each member in the community (thus, questions about "ability-to-pay" (Littlechild (1975)), or "willingness to pay" (Sharkey (1982a)) can be addressed within the model); and (b) is general enough so that it can be applied to many situations (e.g., multi-product firms, publicly owned or regulated enterprises, etc.). Section 3 considers the model as an n-person cooperative game. In Section 4, core-allocations are characterized relative to their implications for efficiency and stability of outputs, cost allocation schemes and industrial structures. The same section addresses questions of "sustainability" of particular market structures. Section 5 utilizes an intuitive notion of cross-subsidization to show that the existence of core-allocations is neither a necessary nor a sufficient condition for the existence of subsidy-free cost allocation schemes. A definition of cross-subsidization is proposed in Section 6. In particular, we use some of the previous arguments (from Section 5) to introduce two axioms that will make any notion of cross-subsidization "precise." This leads to a series of conditions that any "precise" and "complete" notion of cross-subsidization should not violate. Finally, in the same section, we utilize the principle that a cost allocation scheme can be considered subsidy-free only if, whenever possible, those who benefit from the service cover its costs, in order to set up one of the weakest tests of cross-subsidization that will satisfy our conditions.

2. A Model for the Provision of a Service to a Community

We are interested in a community whose members will be indexed by the finite set $N = \{1,\ldots,n\}$. N can be interpreted, alternatively, as the set of individuals, households, localities, neighborhoods, etc., in the community.

This is a single-period model, where I^i will represent the nonnegative income of the i-th member in the community, $i \in N$. Each member in this community derives utility from the consumption of two commodities in quantities x^i and y^i, respectively. Commodity x is a composite commodity. We are not going to specify what commodity y is, but we shall refer to it, from now on, as a service. One can think of it as representing: (a) the service of a "public enterprise" such as electricity, water, natural gas, and telephone, or (b) an excludable public good, or (c) a differentiated product (differentiated as to neighborhoods, localities, groups of users, etc.), or (d) the product of a multi-product enterprise, or even (e) an ordinary private good.

For each $i \in N$, we shall denote by $u^i(x^i,y^i)$ that member's utility function, and we assume that $u^i(.)$ is an increasing function of x^i and a non-decreasing function of y^i.

Any quantity of the composite commodity can be purchased freely by each individual member in the community at the price of one unit of income per unit of that commodity. However, initially, in the community that we examine, there are no enterprises that will provide the service on demand and special arrangements have to be made by the community for that purpose. Nonetheless, each group $S \subseteq N$, of size s, i.e., $S = \{1,\ldots,s\}$, can make its own, separate, arrangements for the provision of a vector $y^S = (y^1,\ldots,y^s)$ of the service to its members.[2]

Although a production process for the service could be specified, here we shall deal directly with costs. In terms of units of income, $C(y^S;S)$ will denote the cost that must be borne by group S if that group, acting unilaterally, is to provide its members with the vector of the service y^S.[3] We make no assumption about subadditivity of costs.[4] It is assumed that: (a) $C(y^S;S) > 0$, $\forall\, S \subseteq N$, if y^S is such that a positive quantity of the service is provided to some member in S, and (b) $C(y^S;S) = 0$, $\forall\, S \subseteq N$, if $y^S = 0$.

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[2]This assumption can be interpreted as saying that each group $S \subseteq N$ has the technology to produce vectors y^S of the service for its members. Thus, the model can be seen as a model of "voluntary clubs." Alternatively, we can follow Sharkey (1982b) and say that there is a "pool of potential firms" each of which can be commissioned by a different group in the community to provide the members of the latter with the service under specified terms. In turn, those terms may include a fixed rate of profit for those firms.

[3]The inclusion of the variable S in the cost function indicates that group composition (including size) may be an important determinant of costs. For example, in addition to costs and benefits that may result from group size, some services may involve costs or savings relative to location, distances, etc., specific to the particular members included in a group.

[4]A cost function is subadditive if and only if $C(y^S;S) + C(y^{S*};S*) \geq C(y^{(S \cup S*)};(S \cup S*))$, for any $S, S* \subseteq N$ such that $S \cap S* = \emptyset$, and for any y^S and y^{S*}. Although our model will be useful only if there are some benefits from cooperation, at least to some groups, these benefits do not have to extend to the union of every disjoint pair of groups in order for our analysis to hold.

For a group $S \subseteq N$ that chooses to act independently, a consumption vector, of the composite commodity and the service, written as a pair (x^S, y^S), is attainable if and only if

(1) $$x^i \geqq 0,\ y^i \geqq 0,\ \forall\ i \in S, \text{ and } \sum_{i \in S} I^i \geqq C(y^S;S) + \sum_{i \in S} x^i,$$

Let us observe that by not assuming subadditivity of costs, what is attainable by the community as a whole may not be the same as what is attainable by the community acting as a single group for the provision of the service. Thus, let us denote by the Greek letter τ a partition of the community into disjoint groups, and let us denote by T the set of all such partitions. An allocation of the composite commodity and the service, in the community, consists of a pair (x,y) where $x = (x^1,\ldots,x^n)$ and $y = (y^1,\ldots,y^n)$. An allocation (x,y) is feasible if and only if

(2) $$x^i \geqq 0,\ y^i \geqq 0,\ \forall\ i \in N, \text{ and}$$

$$\sum_{i \in N} I^i \geqq \min_{\tau \in T} \{ \sum_{S \in \tau} C(y^S;S) \} + \sum_{i \in N} x^i,$$

where y^S is the subvector of elements in y that correspond to the members of group S. Our previous observation, then, is equivalent to saying that allocations (x,y) that satisfy (2) will not necessarily satisfy (1) with respect to N.

Let Y denote the set of all feasible allocations for the community. Then, Y^*, where

(3) $$Y^* = \{(x,y) \in Y: \sum_{i \in N} I^i = \min_{\tau \in T} \{ \sum_{S \in \tau} C(y^S;S) \} + \sum_{i \in N} x^i\}$$

will represent the set of efficient allocations.

The actual costs realized by the entire community for obtaining a vector of the service y, independently of what arrangements have brought that vector, will be denoted by $C(y)$, $C(y) \geqq 0$. A cost allocation scheme relative to $C(y)$, denoted by $c(y)$, consists of a distribution of the total actual costs $C(y)$ among the members of the community. Thus,

$$(4) \qquad c(y) = (c^1(y),\ldots,c^n(y))$$

where

$$(5) \qquad \sum_{i \in N} c^i(y) = C(y).$$

Assuming that the community must finance on its own the costs of the service, we can restrict our attention to feasible allocations $(x,y) \in Y$, and cost allocation schemes $c(y)$ such that

$$(6) \qquad \sum_{i \in N} I^i \geqq \sum_{i \in N} c^i(y) + \sum_{i \in N} x^i.$$

Furthermore, within these bounds of feasibility, and under the same assumption, there is a one to one correspondence between an efficient allocation (x^*,y^*) and a choice of (a) the vector of the service y^*, and (b) the cost allocation scheme $c^*(y^*)$ that lead to this allocation. In other words, given an efficient allocation (x^*,y^*) there corresponds a unique cost allocation scheme $c^*(y^*)$ consistent with this allocation (i.e., where the community finances on its own the service) given by

$$(7) \qquad c^{*i}(y^*) = I^i - x^{*i}, \text{ for each } i \in N.$$

Conversely, given y^*, and given that $\sum_{i \in N} I^i \geqq C(y^*)$, the only cost allocation scheme that leads to an efficient allocation $(x^*,y^*) \in Y^*$, is the one given by (7). In both cases we can check that the respective cost allocation scheme $c^*(y^*)$ must satisfy

(8) $$\sum_{i \varepsilon N} c^{*i}(y^*) = \underset{\tau \varepsilon T}{\text{Min}} \; \{ \sum_{S \varepsilon \tau} C(y^{*S};S)\}.$$

An implication of these conclusions is the following. For any allocation $(x^*,y^*) \varepsilon Y^*$ we can always find out what the corresponding cost allocation scheme is from (7). Conversely, given y^*, and given a cost allocation scheme $c^*(y^*)$ that satisfies (8), and such that $\sum_{i \varepsilon N} I^i \geqq \sum_{i \varepsilon N} c^{*i}(y^*)$, we can always find the corresponding efficient allocation (x^*,y^*), again from (7).

Another implication of the above conclusions is that we can extend the notion of efficiency applied to allocatins in the set Y^*, to hold for the corresponding cost allocation schemes. Thus, for a given vector of the service y^*, we will call a cost allocation scheme $c^*(y^*)$ efficient if and only if the corresponding consumption vector x^* obtainable from (7) leads to an allocation $(x^*,y^*) \varepsilon Y^*$.

Let us observe now that an efficient cost allocation scheme $c^*(y^*)$ must satisfy (8) relative to y^*. Furthermore, in our model (see footnote 2) a partition τ of the community into disjoint groups, for obtaining a vector y^* of the service, can be interpreted as representing the industrial structure that provides that vector of the service to the community. In other words, given τ, we can identify each $S \varepsilon \tau$ with a firm that serves exclusively the members of S. Therefore, we can refer to any partition τ^* that solves the minimization problem,

(9) $$\underset{\tau \varepsilon T}{\text{Minimize:}} \; \sum_{S \varepsilon \tau} C(y^S;S), \text{ given } y,$$

as representing an efficient industrial structure for the provision of the vector y of the service to the community.[5]

Assuming that a positive amount of the service is provided to each member in the community, there is a vector of implicit "prices" that corresponds to each cost allocation scheme. For example, if $y = (y^1,\dots,y^n)$ represents a vector of the service, $y^i > 0$, $\forall\ i \in N$, and if $c(y)$ is a cost allocation scheme, then, $p(y)$, where

$$(10) \qquad p(y) = (p^1(y).,,,p^n(y))$$

and where,

$$(11) \qquad p^i(y) = \frac{c^i(y)}{y^i}, \text{ for each } i \in N,$$

will represent this vector of implicit "prices." However, we should point out that, in our model, a cost allocation scheme may represent a vector of fixed changes determined through some bargaining process among the members of the community. Therefore, the term "prices" may be inappropriate in our model. Only ex post we can infer to a $p(y)$ as prices. Elsewhere in the literature (see, e.g., Faulhaber (1975)), a vector of implicit prices $p(y)$ is called a "price structure."

Because a vector of implicit prices is derivable directly from a cost allocation scheme, we will refer to a $p(y)$, whenever defined, as an efficient vector of implicit prices or an efficient price structure, whenever the corresponding cost allocation scheme $c(y)$ is efficient.

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[5]Note that a solution to (9) depends on the particular vector of the service y. Therefore, it is not necessary that the same industrial structure should be efficient independently of y.

Finally, let us conclude this section by observing that an allocation (x,y) is Pareto optimal for the community if and only if: (a) $(x,y) \in Y$, i.e., it is feasible, and (b) there does not exist a feasible allocation $(x^*,y^*) \in Y$ such that $u^i(x^{*i},y^{*i}) \geqq u^i(x^i,y^i)$, $\forall\, i \in N$, and for at least one $j \in N$, $u^j(x^{*j},y^{*j}) > u^j(x^j,y^j)$. However, each utility function $u^i(\cdot)$ has been assumed to be increasing in at least the variable x^i. Therefore, any Pareto optimal allocation is efficient, that is, it belongs to the set Y^*.

3. The Model as an n-Person Cooperative Game

The model presented in the preceding section is equivalent to an n-person cooperative game where the set N represents the set of players, each subset $S \subseteq N$ represents a coalition, and an alignment of the members of the community into a set of disjoint groups τ represents a coalition structure. In terms of consumption vectors, what each coalition in this game can attain for its members, if it acts independently, can be represented by the set of such vectors that satisfy condition (1). We shall denote this set by Y^S, for each $S \subseteq N$. Thus,

$$Y^S = \{(x^S,y^S): \text{ condition (1) is satisfied}\}. \tag{12}$$

Let us observe now that for each coalition $S \subseteq N$, and for each consumption vector $(\bar{x}^S,\bar{y}^S) = ((\bar{x}^1,\bar{y}^1),\ldots,(\bar{x}^s,\bar{y}^s))$ attainable by that coalition, i.e., for each $(\bar{x}^S,\bar{y}^S) \in Y^S$, there corresponds a vector of utilities $\bar{u}^S = (\bar{u}^1,\ldots,\bar{u}^s)$, where, for each $i \in S$, $\bar{u}^i = u^i(\bar{x}^i,\bar{y}^i)$. Therefore, for each coalition $S \subseteq N$, there is a set of attainable utility vectors associated with the set Y^S that we shall denote by $V(S)$. Thus,

$$V(S) = \{\bar{u}^S = (\bar{u}^1,\ldots,\bar{u}^s): \ (\bar{x}^S,\bar{y}^S) = ((\bar{x}^1,\bar{y}^1),\ldots,(\bar{x}^s,\bar{y}^s)), \ (\bar{x}^S,\bar{y}^S) \in Y^S, \text{ and } \bar{u}^i = u^i(\bar{x}^i,\bar{y}^i), \text{ for each } i \in S\}. \tag{13}$$

In game-theoretic terms, then, we can say that our model is equivalent to an n-person cooperative game in characteristic function form (N,V), where V is the set valued function defined for each $S \subseteq N$, by (12), and it represents the <u>characteristic function</u> of the game.[6]

Solution concepts for a game (N,V) are defined over the space of attainable utility vectors for the set of players N, that is, the set of utility vectors that correspond to the set of feasible allocations Y. However, because our interest here lies with the set of allocations that correspond to such utility vectors, we will proceed to examine directly such allocations. In particular, here we shall examine the set of allocations corresponding to the core of the game (N,V), that is, the <u>set of core allocations</u> for the community.

Thus, let $(\tilde{x},\tilde{y}) = ((\tilde{x}^1,\tilde{y}^1),\ldots,(\tilde{x}^n,\tilde{y}^n))$ and $(\overline{x},\overline{y}) = ((\overline{x}^1,\overline{y}^1),\ldots,(\overline{x}^n,\overline{y}^n))$ be any two feasible allocations in the set Y, and let $(\tilde{x}^S,\tilde{y}^S) = ((\tilde{x}^1,\tilde{y}^1), \ldots,(\tilde{x}^s,\tilde{y}^s))$ be the subvector of elements of $(\tilde{x},\tilde{y})$ corresponding to the members of some coalition $S \subseteq N$. We shall say that <u>the allocation</u> $(\tilde{x},\tilde{y})$ <u>dominates the allocation</u> $(\overline{x},\overline{y})$ <u>via coalition</u> S, and we shall write $(\tilde{x},\tilde{y})$ $\underline{\text{dom}}_S$ $(\overline{x},\overline{y})$, if and only if (a) $S \neq \emptyset$, (b) $(\tilde{x}^S,\tilde{y}^S) \in Y^S$, i.e., $(\tilde{x}^S,\tilde{y}^S)$ is attainable for coalition S, (c) $u^i(\tilde{x}^i,\tilde{y}^i) \geqq u^i(\overline{x}^i,\overline{y}^i)$, $\forall\ i \in S$, i.e., the allocation $(\tilde{x},\tilde{y})$ provides each member of coalition S at least the same level of utility as the allocation $(\overline{x},\overline{y})$, and (c) $u^j(\tilde{x}^j,\tilde{y}^j) > u^j(\overline{x}^j,\overline{y}^j)$, for some $j \in S$, i.e., at least one member j of coalition S is better

[6] In the game (N,V) it is not assumed that utility is transferable. However, a cooperative game with transferable utility could have been obtained from our model, say the game (N,v), by postulation that, for each $S \subseteq N$, v(S), where $v(S) = \underset{\overline{u}^S \in V(S)}{\text{Max}} \{ \sum_{i \in S} \overline{u}^i \}$, represents the <u>value</u> of that coalition. In general, these two games will not be equivalent because utility vectors that are attainable under the game (N,v) may not be attainable under the game (N,V).

off with the consumption bundle $(\tilde{x}^j,\tilde{y}^j)$ than what he is with the consumption bundle $(\bar{x}^j,\bar{y}^j)$. We shall say that the allocation $(\tilde{x},\tilde{y})$ dominates the allocation $(\bar{x},\bar{y})$, (without reference to any specific coalition), and we shall write $(\tilde{x},\tilde{y})$ $\underline{\text{dom}}$ $(\bar{x},\bar{y})$, if and only if there exists a non-empty coalition $S \subseteq N$ such that $(\tilde{x},\tilde{y})$ $\underline{\text{dom}}_S$ $(\bar{x},\bar{y})$.

The set of core-allocations for the community, denoted by the letter K, consists of all feasible allocations that are not dominated by other feasible allocations, i.e.,

$$(14) \qquad K = \{(x,y) \varepsilon Y: \ \nexists \ (\tilde{x},\tilde{y}) \ \varepsilon \ Y, \ni: \ (\tilde{x},\tilde{y}) \ \underline{\text{dom}} \ (x,y)\}.$$

In different terms we can say that the set of core-allocations for the community consists of all those feasible allocations such that no group in the community, acting unilateraly, can attain an alternative consumption vector that will make at least one of the members better off while all other members of the group remain at least indifferent. In this sense then, we can say that a core allocation is stable because no coalition of players has the incentive to depart from it. From the definition of the binary relationship dom, above, and the definition of Pareto optimality in the preceding section, it is obvious that core allocations, if any, are Pareto optimal, and thus, efficient. Therefore, $K \subseteq Y^*$.

4. Core Allocations and the Provision of a Service

Assuming that for the community under consideration the set of core allocations K is not empty (something that is not always guaranteed) we can draw several conclusions about the provision of the corresponding vectors of the service. That is, conclusions about cost allocation schemes, price structures, and industrial structures.

Because core-allocations are efficient, i.e., $K \subseteq Y^*$, we can utilize (7) to find the corresponding, to each element in K, cost allocation scheme for the service. Let,

$$\text{(15)} \qquad KC = \{c(y) = (c^1(y),\ldots,c^n(y)): \ (x,y) \in K, \text{ and } c^i(y) = I^i - x^i, \text{ for each } i \in N\}.$$

Then, KC represents all cost allocation schemes for the service corresponding to the set of core-allocations K. From now on, we shall refer to KC as the set of core cost allocation schemes for the service in the community.

In a similar manner, and provided that for each allocation $(x,y) \in K$, $y^i > 0$, $\forall\, i \in N$, we can utilize (11) to obtain the set of core price structures for the service, KP, where,

$$\text{(16)} \qquad KP = \{p(y) = (p^1(y),\ldots,p^n(y)): \ c(y) \in KC, \text{ and } p^i(y) = \frac{c^i(y)}{y^i}, \text{ for each } i \in N\}.$$

What does it mean for a cost allocation scheme c(y) to belong to the set KC (or the corresponding price structure p(y) to belong to the set KP)? In the first place, it means that the corresponding vector of the service y has been obtained by the community efficiently, that is at the minimum total cost possible, which follows from the efficiency of core allocations. Thus, in the terminology of Section 2, the elements of the set KC are efficient cost allocation schemes, and the elements of the set KP are efficient price structures. Furthermore, this notion of efficiency extends to the coalition structures formed by the community in obtaining the service. Thus, for each cost allocation scheme $c(y) \in KC$, there exists some coalition structure τ^* that solves the minimization problem in (9), that represents the industrial structure through which the community can obtain the corresponding vector of the service y efficiently.

However, we should point out that the existence of core-allocations is not a necessary condition for obtaining efficiency of (a) cost allocation schemes, (b) price structures, and (c) industrial structures. One could obtain such efficiency directly from an allocation in the set Y*. It is the stability properties of core-allocations, then, that is more important, In particular, because, as we have seen in the preceding section, no coalition has the incentive to move away from a core-allocation $(x,y) \in K$, the corresponding cost allocation scheme $c(y) \in KC$ (or price vector $p(y) \in KP$) will also be stable. In the context of our model, this means that no group $S \subseteq N$, acting unilateraly, can obtain its own consumption vector y^S, corresponding to a vector of the service y such that $c(y) \in KC$, at a lower cost (or a lower price).

To prove this last statement, let $c(y) \in KC$, and let (x,y) be the corresponding allocation in the set K. A group in the community, $S \subseteq N$, $S \neq \emptyset$, that acts unilateraly in obtaining y^S must bear a cost equal to $C(y^S;S)$. If the group could obtain y^S, on its own, at a lower cost than that corresponding to $c(y)$, then, $\sum_{i \in S} c^i(y) > C(y^S;S)$. Thus, $\sum_{i \in S} I^i - C(y^S;S) > \sum_{i \in S} (I^i - c^i(y))$, and group S could obtain a higher consumption of the composite commodity for at least some of its members. Because utility functions have been assumed to be increasing in x^i, the allocation (x,y) will be dominated via coalition S. A contradiction, since $(x,y) \in K$.

An important implication of this result is the following. For each $c(y) \in KC$,

$$C(y^S;S) \geqq \sum_{i \in S} c^i(y), \ \forall \ S \subseteq N, \ S \neq \emptyset. \tag{17}$$

In simpler terms, (17) says that each core cost allocation scheme is such that no, non-empty, coalition pays a higher share for the cost of the service than what it would have to pay if it was to provide the same service to its members on its own. However, we should point out that the converse of this statement is not necessarily true. In other words, a cost allocation scheme $c(y)$ that satisfies (17) for some y does not, necessarily, belong to the set KC.[7]

Stability of core allocations and the corresponding core cost allocation schemes (or core price structures) has an important implication about the stability of the corresponding industrial structures that can provide the service to the community. In particular, let us say that, in our model, a coalition structure $\tau \in T$ represents a stable industrial structure for the provision of a vector y of the service to the community if and only if no group in the community has the incentive to move away from that arrangement. It is obvious that any industrial structure that can provide to the community a vector of the service y at a cost allocation scheme $c(y)$ such that the corresponding allocation (x,y) is a core allocation, and thus, $c(y) \in KC$, satisfies the above notion of stability. This, in conjunction with our earlier conclusion about the correspondence of cost allocation schemes $c(y) \in KC$ and coalition structures τ^* that solve (9), and thus, represent efficient industrial structures, makes the proof of the following proposition trivial.

Proposition 1: For each core allocation $(x,y) \in K$, any coalition structure $\tau \in T$ that solves the minimization problem in (9), relative to y, represents a stable industrial structure for the provision of y.

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[7]The set KC has been derived from the set of core-allocations K and not from all those cost allocation schemes that satisfy (17) for various vectors of the service y. Thus, a cost allocation scheme could satisfy (17), relative to some y, and yet the resulting allocation might not belong to the set K because it is not only the cost function that determines K, but preferences as well. However, as we shall see below, under some special assumptions about utility and cost functions, every cost allocation scheme that satisfies (17) will also belong to the set KC.

Let us observe that, like with the case of efficiency, stability of industrial structures depends on the corresponding vector of the service y to some allocation $(x,y) \varepsilon K$. In othe words, for different allocations in K (different as to y) we could have different coalition structures representing stable industrial structures (in addition to possible variations with respect to a specific $(x,y) \varepsilon K$). Suppose, however, that, without going into the specifics that this holds, we were to assume the following. Within the bounds of feasibility in (2), the solution(s) to the minimization problem in (9) is (are) independent of y. In other words, let us suppose that any coalition structure $\tau \varepsilon T$ that represents an efficient industrial structure for the provision of a vector y of the service, does so with respect to any efficient y. Then, Proposition 1 will take the following form.

Proposition 2: If (9) has a solution which is independent of y, any coalition structure that solves (9) represents a stable industrial structure for the provision of the respective vector of the service in any $(x,y) \varepsilon K$.

Proposition 2 brings us to the question of "sustainability" of particular market forms, such as monopoly, duopoly, oligopoly, etc. (On the subject, see, in particular, Sharkey (1982b).) Within the framework of our model, a particular market form of a given composition will be revealed through the coalition structures formed by the community in obtaining the service. In particular then, such market forms, will be sustainable if and only if the corresponding coalition structures represent a unique, stable, structure for the industry.

Utilizing the stability properties of the core, we can obtain the following corollary to Proposition 2.

Corollary 1: If $K \neq \emptyset$, a coalition structure $\tau^* \in T$ represents a sustainable market form for the provision of the service if and only if it is a unique solution to (9), independently of y.

Let us observe that a solution to (9) should be independent of y is a necessary condition for sustainability for the following reason. The core may contain many allocations and each one of those allocations can be used as a, potential, stable outcome. Therefore, if a solution to (9) is not independent of y, then, according to Proposition 1, different coalition structures could represent stable industrial structures for different y's, and none of them will be sustainable. With this observation in mind the proof of Corollary 1 follows directly from (a) the definition of sustainability, (b) the assumption that the core is not empty, and (c) Proposition 2.

As a consequence of Corollary 1, if the core is not empty, a natural monopoly will be sustainable because the coalition structure $\tau^* = \{N\}$ is the unique solution to the minimization problem in (9) independently of y, which is what we mean by natural monopoly.

Questions regarding the choice of the core as a criterion for stability aside, there can be no doubt about the implications of core allocations, in the provision of a service discussed so far.[8] However, this may not be the case regarding issues of cross-subsidization and the relationship of the core to such issues. Because the subject deserves special attention we examine it separately in the next section.

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[8]The results obtained in this section with respect to the core, will hold equally well if we were to use a different solution concept to the respective cooperative game that selects efficient allocations. The only difference will be on the sense of stability utilized in each case.

5. Core Allocations and Issues of Cross-Subsidization

Faulhaber (1975), in examining the issue of cross-subsidization in "public enterprises," proposed that the core of an appropriately defined cooperative game should be utilized as defining "subsidy-free" prices for the respective services. This approach has been adopted by other authors as well (see, e.g., Sharkey (1982a,b)).

Simply put, in our model, a cost allocation scheme c(y), or the corresponding price structure p(y), will result in cross-subsidization if and only if (to paraphrase Faulhaber (p. 966)), it "unduly" favors one group in the community at the expense of another group. Therefore, according to Faulhaber's proposal, cost allocation schemes $c(y) \in KC$ are subsidy-free, thus, they do not unduly favor one group in the community at the expense of another group. Furthermore, for any cost allocation scheme $c(y) \notin KC$, we should be able to find some group in the community that it is unduly favored at the expense of another group, and thus, a case where the latter subsidizes the former. In essence, then, what Faulhaber's proposal amounts to, in our model, is the following. The existence of core cost allocation schemes should be viewed as a necessary and sufficient condition for the existence of subsidy-free cost allocation schemes.

Although an expression such as "unduly favors" is vague and subject to a number of interpretations, there seems to be some confusion between stability, on one hand, and cross-subsidization, on the other. The core does, in the sense explained in the last section, yield stable cost allocation schemes. However, it is doubtful whether all core cost allocation schemes are subsidy-free. Conversely, there is no reason to believe that a cost allocation scheme must be stable in order to be subsidy-free.

Let us examine the second point first, by using a similar example to that of Faulhaber.

Example 1: A community consisting of three neighborhoods, i.e., $N = \{1,2,3\}$, examines the possibility of supplying each neighborhood with water. Any single neighborhood can make its own arrangements for providing itself with up to 10,000 gallons of water at a fixed cost of \$300. Any two neighborhoods can get together and obtain up to 20,000 gallons of water for common use at a fixed cost of \$350. Finally, all three neighborhoods, acting jointly can obtain up to 30,000 gallons of water for common use at a fixed cost of \$600. Additional water can be obtained through idential to the above arrangements and under the same conditions. It is assumed that all three neighborhoods are idential in all respects. In particular, $I^i = \bar{I} \geqq \$300$, $\forall\ i \in N$, while the representative utility function for each neighborhood $i \in N$, between consumption of water y^i and the composite commodity x^i takes the following form: $u^i(x^i,y^i) = x^i + 2(y^i) - 0.0001(y^i)^2$, if $x^i \geqq 0$ and $0 \leqq y^i \leqq 10{,}000$, while $u^i(x^i,y^i) = x^i + 10{,}000$, if $x^i \geqq 0$ and $y^i > 10{,}000$.

It is easy to check that for this example, Pareto optimality requires allocations (x^*,y^*) such that:

$$(18) \qquad \sum_{i \in N} x^{*i} = 3\bar{I} - 600,$$

and

$$(19) \qquad y^* = (10{,}000,\ 10{,}000,\ 10{,}000).^{9}$$

[9] If (18) is violated, and more than \$600 are spent for water someone receives no utility by consuming water above 10,000 gallons, and his position can be improved by consuming more of the composite commodity and less water. The converse holds if less than \$600 are spent for water. With (18) satisfied, (19) is obvious.

Because these allocations can be obtained only through the cooperation of all neighborhoods, the grand coalition represents the most efficient coalition structure for providing the vector of water y^* in (19). Furthermore, with the optimum supply of water for each neighborhood fixed at y^*, the only question that remains to be answered is how the community will distribute the cost of \$600 among the three neighborhoods. This means that, here, we deal with a special case where the satisfaction of (17) by a cost allocation scheme $c(y^*)$, that also satisfies (18) will be a necessary and sufficient condition for the existence of allocations in the core K and core cost allocation schemes in KC.

It is easy to check that no cost allocation scheme will satisfy these conditions for this particular example, and thus K and KC are empty sets. As a consequence, here we deal with a case very similar to that examined by Faulhaber (1975, p.974), where an empty core is interpreted to imply that any cost allocation scheme associated with y^* will involve cross-subsidization. (See also a similar example and interpretation in Sharkey (1982a, p. 58).)

Suppose now that the community was to select the vector y^* and the cost allocation scheme $\bar{c}(y^*) = (200, 200, 200)$ which satisfies (18). Although this cost allocation scheme does not satisfy the stability properties of the core, we cannot say that it "unduly favors one group in the community at the expense of another group." Certainly, every 2-neighborhood coalition pays, at $\bar{c}(y^*)$, \$50 more than what it would have to pay if it was to make its own arrangements for providing 10,000 gallons of water to each of its members. However, this is true for <u>every</u> 2-neighborhood coalition, and not just, say, neighborhoods 1 and 2. On the other hand, if \$50 is the amount by which, say, neighborhood 3 is "unduly favored at the expense of neighborhoods 1 and 2," it is not just neighborhood 3 that it is "favored" in this way, but every neighborhood $i \varepsilon N$. But this is self-contradictory. Stated differently, no matter how vague an expression like "unduly favors" is, one should expect that, in a situation where everyone is identical to the others in all respects, an equal distribution of costs for equal quantities of the service does not favor any group in the community at the expense of another, particularly if such cost allocation scheme is efficient.

What the above arguments amount to is a proof of the following proposition.

<u>Proposition 3</u>: In the provision of a service to a given community, the existence of the core is not a necessary condition for the existence of subsidy-free cost allocation schemes.

Next, let us consider the claim that core cost allocation schemes do not unduly favor one group in the community at the expense of another.

<u>Example 2</u>: A community consisting of five neighborhoods, i.e., $N = \{1,2,3,4,5\}$, faces a similar problem to that of the community in Example 1. In fact, the utility function of each neighborhood in this community is identical to that of each neighborhood in Example 1, while $I^i = \bar{I} \geqq \$500$, $\forall\ i \varepsilon N$. However, the composition of each coalition in determining the relevant costs in this particular community is important. (For example, we can think of the location of water, ground considerations, and distances as playing some role in determining costs.) More specifically the community can be devided into two groups of neighborhoods, say, P and Q, such that cooperation among neighborhoods can be benefitial only if they belong to different groups. In particular, we shall let $P = \{1,2\}$, and $Q = \{3,4,5\}$. Each neighborhood $i \varepsilon N$ can make its own arrangements

for providing itself with $\bar{y}^i$ gallons of water, up to 10,000, at a fixed cost of \$500. Cooperation, for obtaining the same quantity of water for each member of a coalition S, yields the cost function $C(\bar{y}^S;S)$, given by

$$C(\bar{y}^S;S) = 500s - 1000(\min\{|S \cap P|, \tfrac{1}{2}|S \cap Q|\}), \ \forall\ S \subseteq N,$$

where, s denotes the number of neighborhoods in coalition S, and $|S \cap P|$, $|S \cap Q|$, denote the number of neighborhoods in P and Q respectively, that also belong to coalition S. (In essence, the term $1000(\min\{|S \cap P|, \frac{1}{2}|S \cap Q|\})$, in the above cost function, represents the savings from cooperation for each coalition S.) As in Example 1, additional water can be obtained through arrangements identical to the above and under the same conditions.

The same arguments used in discussing Example 1, can be utilized here to show that Pareto optimality requires allocations (x^*,y^*) such that:

$$(20) \qquad \sum_{i \in N} x^{*i} = 5\bar{I} - 1000,$$

and

$$(21) \qquad y^* = (10{,}000,\ 10{,}000,\ 10{,}000,\ 10{,}000,\ 10{,}000).$$

These allocations are obtainable either through the grand coalition N, or through coalition structures $\tau^* = \{S^*,S^{**}\}$, where $S^* = \{i,j\}$, $i \in P$, $j \in Q$, and $S^{**} = (P-\{i\}) \cup (Q-\{j\})$.

With the optimum supply of water fixed at y^*, the only question that remains to be answered is how the community will distribute the cost of \$1,000 among the five neighborhoods. Thus, here we deal again with the special case where the satisfaction of (17) by a cost allocation scheme $c(y^*)$, that also satisfies (20), will be a necessary and sufficient condition for the existence of the core K and core cost allocation schemes in KC.

It is well known (see, e.g., Maschler (1976)) that for a cooperative game like this, the set KC is non-empty and it consists of the single point (500, 500, 0, 0, 0). Thus, the only core cost allocation scheme is the one where 10,000 gallons of water are provided to each neighborhood at a total cost of $1,000 borne, at an equal share, by neighborhoods 1 and 2 only. The corresponding core allocation is $(\hat{x}^*,\hat{y}^*)$, where, $\hat{y}^*$ is given by (21), and $\hat{x}^*$ is such that: $\hat{x}^{*1} = \hat{x}^{*2} = \bar{I} - 500$, while $\hat{x}^{*3} = \hat{x}^{*4} = \hat{x}^{*5} = \bar{I}$.

No matter how vague an expression such as "unduly favors" is, it will be very hard to convince neighborhoods 1 and 2 that the above core cost allocation scheme does not unduly favor neighborhoods 3, 4, and 5. It is a different story to say that neighborhoods 1 and 2 do not have the power to move away from the above core cost allocation scheme (and, thus, that it is stable) and a different story to say that such a scheme does not unduly favor 3, 4, and 5.

To take the argument one step further, why not call the alternative cost allocation scheme $\bar{c}(y^*)$, where,

$$(22) \qquad \bar{c}(y^*) = (-250, -250, 500, 500, 500),$$

"subsidy-free"? This cost allocation scheme will produce the Pareto optimal allocation $(\bar{x}^*,\bar{y}^*)$, where, $\bar{y}^*$ is given by (21), and $\bar{x}^*$ is such that $\bar{x}^{*1} = \bar{x}^{*2} = \bar{I} + 250$, while $\bar{x}^{*3} = \bar{x}^{*4} = \bar{x}^{*5} = \bar{I} - 500$. Although $(\bar{x}^*,\bar{y}^*)$ does not belong to the core, it represents the other extreme of the core allocation $(\hat{x}^*,\hat{y}^*)$ in the following sense. While at $(\hat{x}^*,\hat{y}^*)$ all benefits from cooperation go to the set of neighborhoods Q, at $(\bar{x}^*,\bar{y}^*)$ all benefits from cooperation go to the set of neighborhoods P. But then, if $(\bar{x}^*,\bar{y}^*)$ unduly favors P, so does $(\hat{x}^*,\hat{y}^*)$ with respect to Q. In particular, the neighborhoods in the set Q need the neighborhoods in the set P in order to realize any benefits from cooperation, as much as the former need the latter for the same purpose.

What the above example illustrates, is the validity of the following proposition.

Proposition 4: In the provision of a service to a given community, the existence of the core is not a sufficient condition for the relevant cost allocation schemes to be subsidy-free.

In addition to illustrating the validity of Proposition 4, Example 2 is interesting for the following reason. Even if the core is not empty, we can still find allocations, not belonging to the core, such that, the corresponding cost allocation schemes do not "unduly favor" one group in the community at the expense of another. For example, the cost allocation scheme $\tilde{c}(y^*) = (200, 200, 200, 200, 200)$, where y^* is given by (21), yields the Pareto optimal allocation (x^*,y^*) where x^* is such that $x^{*i} = I - 200$, $\forall\ i \in N$. The only coalitions that violate (17) relative to $\tilde{c}(y^*)$, are three-neighborhood coalitions S^{**} consisting of one neighborhood from the set P and two neighborhoods from the set Q. But, as with the symmetric allocation in Example 1, this is true for every such coalition. Thus, we cannot make a case that a coalition S^{**}, as above, subsidizes its complement $S^* = N - S^{**}$, at $\tilde{c}(y^*)$. As a consequence, it is possible, in a situation like that of Example 2, that: (a) the core is not empty but core cost allocation schemes are not subsidy-free, and (b) there exist allocations outside the core such that no valid case about cross-subsidization can be made for the corresponding cost allocation schemes.

We have seen so far in this section that core stability and absence of cross-subsidization do not necessarily coincide. Now, one may argue tha Faulhaber's proposal to use the core as defining subsidy-free price structures (or cost allocation schemes) has been based on arguments different from stability. Thus, that core-stability is coincidental to this definition. There are three cases in Faulhaber's paper where this could be the case, but later on the relevant argument is abandoned and thus, it is core stability that he uses in defining subsidy-free price structures.

The first case is where Faulhaber gives the following definition, as a "first approximation," for subsidy-free price structures: "If the provision of any commodity (or group of commodities) by a multicommodity enterprise subject to a profit constraint leads to prices for the other commodities no higher than they would pay by themselves, then the price structure is subsidy-free" (1975, p. 966). Applying this definition to Example 2, we can see that both the core cost allocation scheme (500, 500, 0, 0, 0), and the cost allocation scheme $\bar{c}(y^*)$ given by (22) qualify as "subsidy-free" schemes. For example, $\bar{c}(y^*) = (-250, -250, 500, 500, 500)$ could result as follows. Initially, an enterprise comes into agreement with the neighborhoods in the set Q to provide each of them with 10,000 gallons of water with a zero profit constraint. This constraint will not be violated later on if the same enterprise comes into agreement with the neighborhoods in the set P as well, provide each with 10,000 gallons of water and pass the savings of \$500 to them. (Presumably, the enterprise saves this amount of money by, say, utilizing the location of these neighborhoods.) Although both of the above cost allocation schemes satisfy the above definition, only the core cost allocation scheme is considered subsidy-free by Faulhaber.

To examine the second case, let us consider cost allocatin schemes that satisfy (8) and the following condition.

$$(23) \qquad \sum_{i \in \bar{S}} c^i(y) \geq \underset{\tau \in T}{\text{Min}} \left\{ \sum_{S \in \tau} C(y^S;S) \right\}$$

$$- \text{Min}\left\{ \sum_{S \in (\tau - \bar{S})} C(y^S;S): \ \tau \in T, \bar{S} \in \tau \right\}, \ \forall \ \bar{S} \subseteq N.$$

In our model, (23) represents a general form of the generalized incremental cost test (see, Schotter and Schwodiauer (1980, pp. 489-490) who attribute this term to Faulhaber and Zajac). Now, it so happens that, in certain cases, cost allocation schemes c(y) that satisfy (8) and (17) also satisfy (23). Arguments about its validity as a test defining subsidy-free cost allocation schemes aside, if (a) (17) is a necessary and sufficient condition for a cost allocation scheme to belong to the set KC, and (b) every cost allocation scheme that satisfies (8) and (17) also satisfies (23), one can argue that core cost allocation schemes are subsidy-free because they satisfy the above generalized incremental cost test. However, even this test, which is more general and independent of stability, is rejected by Faulhaber as defining subsidy-free cost allocation schemes in favor of core-stability. As he states, "... there may be prices which pass an incremental cost test and yet involve cross-subsidy" (1975, p. 976).

Finally, the third case is very similar to the above, and involves condition (17). By itself, condition (17) does not define the set of core cost allocation schemes (as we have noted in Section 4). However, it represents the generalized stand above test (see, the same reference as for generalized incremental cost test) for cost allocation schemes, in that each coalition should not have to pay a higher cost for its own consumption vector of the service than what that coalition must pay if it was to act alone. Notice, though, that, in certain cases, this test can be identical to the generalized incremental cost test in (23), which, as we have noted, it has been rejected by Faulhaber as a general test defining subsidy-free price structures, in favor of the core. Thus, (17) is not sufficient, according to these arguments, for the definition of subsidy-free price structures either.

What these conclusions indicate is that if we are to follow Faulhaber's approach we must identify absence of cross-subsidization with stability, and in particular with core-stability. However, as we have seen in deriving Propositions 3 and 4, this is not necessarily the case.

6. What is Cross-Subsidization?

Propositions 3 and 4, and the arguments used to demonstrate their validity leave open the question of how one can define, in a precise way, what cross-subsidization is. In particular, if core-stability is not sufficient to determine whether a cost allocation scheme is subsidy-free, and if an expression like "unduly favors" is vague, what is the alternative? In this section we shall approach the subject from a different perspective.

First of all we should make clear that our purpose here is not to find out whether the community will choose one allocation over another. Neither to say that one allocation is "better" than another. Our task is to characterize, in as a precise way as possible, different cost allocation schemes associated with a vector of the service from the perspective of whether or not they involve some form of cross-subsidization. Thus, our task should not be seen much different than, say, someone trying to determine whether a cost allocation scheme is efficient.

Given a vector of the service y, let us consider an arbitrary cost allocation scheme c(y). How can one go ahead and determine whether c(y) is or is not subsidy-free? Well, we can start out by trying to classify each member of the community in accordance to whether that member is the recipient of a subsidy, a subsidizer, or neither. In particular, let $N_R(c(y))$, $N_P(c(y))$, and $N_O(c(y))$ denote these three categories. Then, in the set $N_R(c(y))$ we should include every member of the community that, relative to c(y) and our notion of what cross-subsidization is, receives a subsidy. Likewise, we should include in the set $N_P(c(y))$ every member of the community who is a subsidizer. Finally, $N_O(c(y))$ should contain all members of the community that we cannot classify either as subsidizers or as the recipients of a subsidy.

Now, any notion of cross-subsidization that is not precise will not enable us to make the above classification. For example, if we cannot decide whether a member of the community is a subsidizer or the recipient of a subsidy, relative to a cost allocation scheme c(y), how are we to decide whether c(y) is subsidy-free? Therefore, any precise notion of what cross-subsidization is should always enable us to make the above classificiation.

Two conditions seem essential for this purpose. The first is a separation axiom (Axiom 1, below) that classifies each member of the community in one and only one of the above three categories. The second is a raison d'être axiom (Axiom 2, below) that requires the existence of subsidizers if there exist recipients of a subsidy and vice versa.

Axiom 1: The three sets $N_R(c(y))$, $N_P(c(y))$, and $N_O(c(y))$ are pairwise disjoint for any cost allocation scheme c(y).

Axiom 2: $(N_R(c(y)) \neq \emptyset) \Leftrightarrow (N_P(c(y)) \neq \emptyset)$, for any cost allocation scheme c(y).

There are two immediate implications that follow from these axioms. The first one is that $N_R(c(y)) \neq N$, and $N_P(c(y)) \neq N$, for any cost allocation scheme $c(y)$. This means that no precise notion of cross-subsidization should classify every member of the community as the recipient of a subsidy, and, likewise, it should not classify every member of the community as a subsidizer. The second implication is the following definition. A cost allocation scheme $c(y)$ is <u>subsidy-free</u> if and only if every member of the community can be classified as belonging to the set $N_0(c(y))$.

It is worth noting here that this definition is independent of any specific notion of what cross-subsidization is. In other words, as long as a notion of what cross-subsidization is satisfies the two axioms (and thus, according to our terminology, is precise) there is only one way to determine when a cost allocation scheme is subsidy-free, and that is through the condition $N_0(c(y)) = N$. As a consequence of this, it may be possible to come up with a variety of notions about cross-subsidization. In other words, certain cost allocation schemes can be characterized subsidy-free in one sense but not under a different sense. Although both notions can be precise.

The natural question to ask here is whether any of the classical tests satisfy the above axioms, and if not whether there exist alternative tests that do.

Proposition 3 suggests that core-stability, the generalized stand alone test in (17), and the generalized incremental cost test in (23), will all violate at least one of the two axioms. The problem with these tests is that they fail to take into account that in certain situations, it will be impossible to separate the subsidizers from the recipients of a subsidy. A case in point is the symmetric cost allocation scheme $\bar{c}(y^*) = (200, 200, 200)$ in Example 1. In this situation, all members of the community are identical in all respects, they receive the same quantity of the service, they share equally the costs of obtaining efficiently that service, the resulting allocation is Pareto optimal, and yet $\bar{c}(y^*)$ would not be considered subsidy-free by any of the three tests mentioned above. But as we have argued in Section 5, it is impossible to separate the subsidizers from the recipients of a subsidy in the above situation. What this means is that any notion of cross-subsidization which is expected to satisfy Axioms 1 and 2 must include some "symmetry condition." That is, a condition which (a) will classify "similar" members of the community as belonging to the same category, i.e., subsidizers, recipients of a subsidy, or neither one of these, and (b) will enable us in situations like the above to classify every member of the community as belonging to the set $N_0(\bar{c}(y^*))$ relative to $\bar{c}(y^*)$.

Althouth some symmetry condition seems necessary if Axioms 1 and 2 are to be satisfied, it is not sufficient for this purpose. A vector of the service y that is not obtained efficiently, can lead to cost allocation schemes with the following characteristic. Some members of the community are classified as subsidizers without being able to identify the recipients of such subsidy. Thus, Axioms 1 and 2 could be violated not because there is cross-subsidization but due to ineffiency. What this conclusion implies is that even the simple "stand along test"

(24) $\qquad c^i(y) \leqq C(y^i;\{i\}), \ \forall \ i \in N,$

will violate Axiom 2 if we do not require that y is obtained efficiently.

A similar situation to this, but in reverse, can arise if we do not require that the community finances on its own the provision of any vector of the service y. In other words, without this requirement we may run into a situation where everyone receives a subsidy without being able to idenfity the members of the community who pay that subsidy. Thus, Axiom 2 could, again, be violated. As a consequence, in a model like the one in Section 2, the satisfaction of Axioms 1 and 2 by any precise notion of what cross-subsidization is, requires not only that a "symmetry" condition should not be violated, but that the "efficiency" and "self-finance" conditions should not be violated either.

Although the above three conditions could make precise a rule for testing cost allocation schemes about cross-subsidization, they are not sufficient for making that rule complete. For example, the simple stand along test in (24) restricted to apply to cost allocation schemes $c(y)$ that correspond to efficient allocations $(x,y) \in Y^*$, will not violate any of these conditions. However, as we have seen in Section 5, a cost allocation scheme can satisfy the so-modified simple stand alone test and still may not, necessarily, be subsidy-free. (As we have argued in deriving Proposition 4, the core cost allocation scheme of Example 2 is not subsidy-free. However, it satisfies the modified simple stand alone test.) What the above conditions do not provide is a fundamental principle that can be used as the basis in any test about cross-subsidization.

To get an idea of what this principle might be, let us recall that we have declared the core cost allocation scheme of Example 2 as not being subsidy-free because one group of the community, the group Q, benefits from the service and they contribute nothing towards the costs of obtaining it. Furthermore, the same members of the community could be contributing something towards the costs of the service, and still they could find the new arrangements preferable to either the non-provision, or the situation where they would have to obtain the service on their own. For example, as we have suggested in Section 5, the cost allocation scheme $\tilde{c}(y^*) = (200, 200, 200, 200, 200)$ represents such alternative. What this suggests is that a condition that requires those members of the community that benefit from the service to contribute towards its costs whenever possible, that we shall refer to as a "pay if you benefit" condition, could serve our purpose. However, the amount of payment may raise different questions (e.g., questions of "fairness").

To summarize the above discussion, core-stability, the generalized stand alone test in (17), the generalized incremental cost test in (23), and the simple stand above test in (24), all violate at least one of Axioms 1 and 2. This has led us to four conditions that any complete and precise notion of what cross-subsidization is, should not violate. In a model like the one presented in Section 2, these are: (a) efficiency, (b) self-finance, (c) symmetry, and (d) pay if you benefit.

Our next task is to make these conditions precise. From the four, efficiency and self-finance will be automatically satisfied if we were to agree that any notion of cross-subsidization that we set forth applies only to cost allocation schemes corresponding to efficient allocation in the set Y^*. As may be recalled from Section 2, the set Y^* has been derived under the assumption of self-finance. In addition, any cost allocation scheme $c(y)$ associated with some allocation $(x,y) \in Y^*$, must satisfy (8), and thus, the efficiency condition will be satisfied as well.

In essence, to agree that any notion of cross-subsidization applies to cost allocation schemes corresponding to allocations in the set Y^*, is equivalent to saying that any other cost allocation scheme will be declared, automatically, as not being subsidy-free. This is a characteristic of our model. For example, in a more general model we could introduce an additional member, say the 0-th member, who represents the "outside world." In such an expanded model, Axioms 1 and 2 will not be violated if we do not utilize the two conditions of efficiency and self-finance. For example, the "outside world" can be classified as a subsidizer if a cost allocation scheme does not satisfy the self-finance condition. On the other hand, the "outside world" can be classified as the recipient of a subsidy if a cost allocation scheme is not efficient. In either a closed model (like the one presented in Section 2 with the efficiency and self-finance conditions), or with a model that includes the "outside world" as a member (but without these conditions), Axioms 1 and 2 can hold. Thus, to agree that any notion of cross-subsidization applies only to cost allocation schemes corresponding to allocations in the set Y*, in a model like that of Section 2, is not as restrictive as it might, originally, sound.

To define a "pay if you benefit" condition, let us consider two alternative choices, for the community. The first one is for the community not to produce any quantity of the service at all and have each member $i \in N$ spend its entire income acquiring $\bar{x}^i$ units of the composite commodity. By so doing each member $i \in N$ will realize the level of utility $u^i(\bar{x}^i,0)$. The second choice is for the community to obtain, through some arrangement (this may include each member acting on its own), a vector of the service y, at the total cost C(y).

To compare the two situations, let us divide the community into the two disjoint groups S^0 and S', where,

$$(25) \qquad S^0 = \{i \in N: \ u^i((I^i-\bar{c}^i),y^i) \leqq u^i(\bar{x}^i,0), \ \forall \ \bar{c}^i > 0\},$$

and S' is the complement of S^0, that is, $S' = N - S^0$. Because $(\bar{x}^i,0)$ is the consumption bundle of each $i \in N$ when the service is not provided at all, we can say that the set S^0 represents all members of the community who cannot benefit from the vector of the service y if they were to contribute any positive amount of money towards its costs. In particular, due to our assumption that $u^i(\cdot)$ is an increasing function of x^i, every member of the community who does not receive any quantity of the service at y will belong to the set S^0. With this interpretation of S^0 in mind, we can say that S' represents all members of the community that, by contributing some positive amount of money towards its costs, they can benefit from the vector of the service y.

With the sets S^0 and S' defined as above, relative to a vector of the service y, and given the total cost C(y), let us consider a cost allocation scheme $\hat{c}(y)$ that satisfies conditions (26) and (27) below.

(26) $$\hat{c}^i(y) \leqq 0,\ \forall\ i \in S^0;\ \hat{c}^i(y) > 0,\ \forall\ i \in S'.$$

(27) $$u^i((I^i - \hat{c}^i(y)),y^i) > u^i(\bar{x}^i,0),\ \forall\ i \in S'.$$

Condition (26) says that at $\hat{c}(y)$, all those members of the community who cannot benefit from the service if they were to contribute any positive amount of money towards its costs, they do not contribute. On the other hand, all those who can benefit from the service even if they contribute a positive amount of money towards its costs, do contribute at $\hat{c}(y)$. Condition (27) says that the contribution of each member of the community in the set S' is not so excessive as to eliminate the benefit. Note that by the definition of a cost allocation scheme (see Section 2) if there exists a $\hat{c}(y)$ that satisfies the above conditions relative to C(y), then $\sum_{i \in S'} \hat{c}^i(y) \geqq C(y)$, that is, the total cost of the service is covered by those in the set S'.

Now, it is not guaranteed that, for each vector of the service y and the corresponding cost C(y), there exist cost allocation schemes that satisfy both conditions (26) and (27). However, the possibility that such cost allocation schemes can exist is sufficient for requiring any complete notion of what cross-subsidization is to satisfy the following <u>pay if you benefit condition</u>.

<u>Condition 1</u>: Relative to a vector of the service y and a total cost C(y), no cost allocation scheme c(y) that violates either one of conditions (26) and (27) should be considered subsidy-free if there exists an alternative cost allocation scheme $\hat{c}(y)$ that satisfies both of these conditions.

It is important to note that Condition 1 <u>does not</u> say that a cost allocation scheme c(y) should be considered subsidy-free if it satisfies both conditions (26) and (27). Neither does it say that if a cost allocation scheme $\hat{c}(y)$ that satisfies both (26) and (27) does not exist we should go ahead and consider any c(y) as being subsidy-free. In either case some other condition might be violated. However, it is possible that a cost allocation scheme c(y) will not violate Condition 1 even if it does not satisfy both conditions (26) and (27). Although, for this to happen there should not exist an alternative cost allocation scheme $\hat{c}(y)$ that satisfies both (26) and (27). On the other hand, if relative to a vector of the service y, (a) there exist cost allocation schemes that satisfy both conditions (26) and (27), and (b) none of our other three conditions is violated, we can consider conditions (26) and (27) as a minimum requirement for a cost allocation scheme to be subsidy-free.

To define a "symmetry" condition we need a notion of "similarity" between any two members of the community. In general terms, we shall say that two members are <u>similar</u> to each other if and only if (a) their income and preferences do not vary widely, and (b) in providing the same quantity of the service to each one of them through some arrangement, one member could replace the other without any effects on the relevant cost. Below we formalize this concept to include both a "strong" and a "weak" sense of similarity.

Let us write S_{ij} to denote a coalition that contains member i but not member j, i.e., $i \in S_{ij}$ and $j \notin S_{ij}$. Let us also write S_{ji} to denote the coalition that will emerge from S_{ij} if we were to replace member i with member j, that is, $S_{ji} = (S_{ij} - \{i\}) \cup (\{j\})$. Formally, any two members in the community i and $j \in N$ are <u>similar to each other (in a strong sense) relative to a vector of the service</u> y if and only if conditions (28a) - (28c), below are satisfied.

(28a) Either $i,j \in S^0$, or $i,j \in S'$, relative to y.

(28b) $y^i = y^j$.

(28c) $C(y^{S_{ij}};S_{ij}) = C(y^{S_{ji}};S_{ji})$, for each pair of coalitions S_{ij} and S_{ji}, as above.

In (28a), S^0 and S' are the same sets utilized in stating Condition 1, above. Thus, that both i and j belong to the same set S^0 or S' is what we mean by saying that there is no wide variation in income and preferences between i and j. In (28c), $y^{S_{ij}}$ and $y^{S_{ji}}$ represent the subvectors of the service corresponding to coalitions S_{ij} and S_{ji}, respectively, given y. To understand (28c) let S be any coalition that contains neither i nor j. If S was to obtain y^S on its own it must bear a cost equal to $C(y^S;S)$. Now suppose that either i or j were to joint that coalition. Then the incremental cost of providing each of them with the same quantity of the service will be $C(y^{S_{ij}};S_{ij}) - C(y^{S};S)$, and $C(y^{S_{ji}};S_{ji}) - C(y^{S};S)$, respectively. By equating the two and canceling out the common term we will obtain the equality in (28c). In essence, then, (28c) says that the incremental cost of providing the same quantity of the service to either i or j by a coalition that, initially, contains neither one of these is the same, and this holds for every such coalition.

Although similarity between any two members of the community, in the above sense, falls short of saying that those members are identical to each other (the latter will require the same utility function and income for each), it is a very strong notion of similarity because the equality in (28c) must hold for each pair of coalitions S_{ij} and S_{ji} with the indicated composition. For some notions of cross-subsidization a weaker condition than (28c) may be more desirable. One alternative, where the incremental cost is measured relative to the entire community and not relative to each pair of coalitions, as above, is the following.

Any two members of the community are <u>similar to each other (in a weak sense) relative to a vector of the service</u> y if and only if conditions (28a), (28b), and (28c'), the last defined below, are satisfied.

(28c') $$\min\{\sum_{S \varepsilon \tau} C(y^{S};S): \tau \varepsilon T, \{i\} \varepsilon \tau, \text{ and } S \neq \{i\}\}$$

$$= \min\{\sum_{S \varepsilon \tau} C(y^{S};S): \tau \varepsilon T, \{j\} \varepsilon \tau, \text{ and } S \neq \{j\}\}.$$

To see why (28c') represents the indicated measurement of incremental cost, let us observe that the minimum total cost for obtaining y is given by $\min_{\tau \varepsilon T} \{ \sum_{S \varepsilon \tau} C(y^{S};S)$. Subtracting each term in (28c') from this minimum total cost will yield the incremental cost of providing each member i and j, respectively, with the quantity y^i, y^j, where $y^i = y^j$, by (28b). By equating the two and canceling out the common term we will obtain (28c'). Thus, (28c') does represent identity of incremental costs between i and j relative to the entire community and relative to the total minimum cost of obtaining y.

It is worth noting here that if all members of the community are identical to each other in all respects (this is the case in Example 1) then, every member of the community is similar to the others relative to a vector of the service where everyone receives the same quantity. On the other hand, in Example 2 where, relative to the vector of the service y* in (21), we can devide the community into the sets P and Q, similarity of members is restricted to within each one of these sets. Finally, it is conceivable that no two members in a community can be considered similar to each other no matter what vector of the service we consider.

By utilizing either notion of similarity, above, we can state our symmetry condition as follows.

Condition 2: For a vector of the service y, and for a cost allocation scheme c(y), suppose that there exists a pair of members $i,j \in N$ that: (a) are similar to each other relative to y, and (b) $c^i(y) = c^j(y)$. Then, relative to c(y): (A) both i and j should be classified as belonging to one and only one of the sets $N_R(c(y))$, $N_P(c(y))$, or $N_0(c(y))$; (B) if (a) and (b) hold for all $i,j \in N$, $i \neq j$, and the efficiency and self-finance conditions are not violated, $N = N_0(c(y))$.

The first part of Condition 2 says that similar members cannot be classified as subsidizing one another if they pay an equal share towards the costs of the service. The second part of the same condition says that if (a) every member of the community is similar to the others and (b) each member pays an equal share towards the costs of the service, the corresponding cost allocation scheme should always be considered subsidy-free, provided that it satisfies the efficiency and self-finance conditions.

Because we have not specified which one of the two notions of similarity (strong or weak) we have used in Condition 2, one can utilize this symmetry condition either in a strong or a weak sense. For that matter, if a notion of what cross-subsidization is, is very strong (thus, the notion of when a cost allocation scheme is subsidy-free, is very weak) it may be irrelevant whether in Condition 2 one uses a strong or a weak notion of similarity.

Having made our four conditions precise, we can proceed to define different notions of cross-subsidization. Because the two conditions of efficiency and self-finance serve only as restrictions, we can take the first step and for each feasible vector of the service y define the set valued function F(y) as follows:

(29) $F(y) = \{c(y)$: (a) condition (8) is satisfied, and
(b) $(x,y) \in Y^*$, where $x^i = I^i - c^i(y)$,
for each $i \in N\}$.

In other words, for each feasible y, F(y) represents all cost allocation schemes c(y) that satisfy the efficiency and self-finance conditions.

We seek to characterize the elements of the set F(y), for each feasible y, as subsidy-free or not, given that Conditions 1 and 2 should not be violated. One way to approach the problem is to combine those conditions with one of (a) core-stability, (b) the generalized stand alone test in (17), (c) the generalized incremental cost test in (23), or (d) the simple stand alone test in (24). Under this approach, each one of these tests could still provide some notion of cross-subsidization assuming that

Conditions 1 and 2 are not violated. For example, a core cost allocation scheme may still be considered subsidy-free, under this approach, if it does not violate Condition 1. On the other hand, even if a cost allocation scheme does not belong to the set KC it could be considered subsidy-free if, under some notion of similarity, we can apply part (B) of Condition 2. However, whether any one of the so-modified tests provides an intuitively appealing notion of cross-subsidization is a different question. In particular, other than the modified core-stability test, the other three do not, necessarily, take into account the preferences of individual members in the community.

An alternative approach is to pursue further the principle of cross-subsidization set forth by the pay if you benefit condition (Condition 1). This means that we should compare any allocations in the set Y* where a positive quantity of the service is provided to some members of the community with the allocation that results from non-provision of the service to any member. Let us then write $(\bar{x},0)$ to indicate this latter allocation. Then, $\bar{x}^i = I^i$, $\forall\ i \in N$, $(\bar{x},0) \in Y^*$, and $F(0) = \{\bar{c}(0)\}$, where $\bar{c}^i(0) = 0$, $\forall\ i \in N$. In other words $(\bar{x},0)$ is efficient, each member's income is spent entirely on the composite commodity, and there is only one cost allocation scheme belonging to the set F(0), where there is no cost borne by any member. The level of utility for each member $i \in N$, at that allocation, is $u^i(\bar{x}^i,0)$. We can then say that if the community does not enter into any agreements for the provision of any positive quantities of the service, and if each member $i \in N$ is left on its own, the minimum level of utility that that member could achieve is $u^i(\bar{x}^i,0)$.

This last conclusion implies that it is reasonable to assume that if the community was to go ahead and obtain a vector of the service y, it will do so if the following condition is not violated.

Rationality in Provision: Relative to a feasible vector of the service y, such that $y^i > 0$ for some $i \varepsilon N$, there exists a cost allocation scheme $c(y) \varepsilon F(y)$ such that $u^i((I^i - c^i(y)),y^i) \geqq u^i(\overline{x}^i,0)$, $\forall\ i \varepsilon N$, and $u^j((I^j - c^j(y)),y^j) > u^j(\overline{x}^j,0)$, for some $j \varepsilon N$.

Simply put, to require the community to obtain vectors of the service y that do not violate the above condition, is equivalent to saying the following. It will be irrational for the community to obtain the service if it is possible that every member will prefer at least as well, if not more, the non-provision to the provision under every cost allocation scheme $c(y) \varepsilon F(y)$. Thus, by ignoring cases where the above condition is violated, is equivalent to saying that we shall not try to apply our notion of cross-subsidization to those cases where the community acts irrationally. Furthermore, we should note that rationality in provision is not necessarily equivalent to individual rationality.[10]

In the remainder of this section we shall undertake a series of steps intended to exclude from the set F(y), for each y that satisfies the Rationality in Provision condition, all cost allocation schemes that involve some form of cross-subsidization. Thus, the elements of F(y) that will not be eliminated through this process can qualify as been subsidy-free.

For each vector of the service y, as above, there always exist cost allocation schemes $c(y) \varepsilon F(y)$ such that

(30) $$u^i((I^i - c^i(y)),y^i) \geqq u^i(\overline{x}^i,0),\ \forall\ i \varepsilon N,$$

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[10] For a model like the one in Section 2, we can say that a feasible vector of the service y and a cost allocation scheme c(y) satisfy the individual rationality condition if and only if $u^i((I^i - c^i(y)),y^i) \geqq \text{Max } V(\{i\})$, $\forall\ i \varepsilon N$, where the sets $V(\{i\})$, $i \varepsilon N$, have been defined in (13).

which follows from the Rationality in Provision condition. Thus, in our process of elimination, we can undertake the first step and, outright, eliminate cost allocation schemes that violate (30) from being subsidy-free.

Step 1: No cost allocation scheme $c(y) \varepsilon F(y)$ that violates (30) should be considered subsidy-free.

The reasoning for undertaking this step is quite simple. If a cost allocation scheme violates (30) some members of the community will prefer non-provision of the service to y, that is, those members' benefits from the service will be negative. Since it will be possible to select an alternative cost allocation scheme where this does not happen, the choice of the former cost allocation scheme over the latter, unduly favors some members of the community at the expense of some other members, and in particular at the expense of those members whose benefit will be negative.

The second step is a direct application of Condition 1, and we state it here for reasons of completeness.

Step 2: No cost allocation scheme $c(y) \varepsilon F(y)$ should be considered subsidy-free if there exists an alternative cost allocation scheme $\hat{c}(y) \varepsilon F(y)$ that satisfies conditions (26) and (27).

The question now is how to characterize cost allocation schemes in $F(y)$ that satisfy (30) but cannot satisfy both conditions (26) and (27), and there is no cost allocation scheme $\hat{c}(y) \varepsilon F(y)$ that will satisfy both of these conditions. We shall approach this problem in a similar way to that of undertaking Step 1, that is, by considering what alternatives are possible.

Given a vector of the service y and a cost allocation scheme c(y), as above, let us suppose that there exists a feasible $\tilde{y}$, $\tilde{y} \neq y$, that satisfies the Rationality in Provision condition, and in addition there exists a cost allocation scheme $\hat{c}(\tilde{y}) \in F(\tilde{y})$ that satisfies (26), (27), and (30). Furthermore, let us suppose that the following condition holds true.

$$(31) \qquad u^i((I^i - \hat{c}^i(\tilde{y})),\tilde{y}^i) \geqq u^i((I^i - c^i(y)),y^i), \; \forall \; i \in N.$$

Then, we can say that $\tilde{y}$ and $\hat{c}(\tilde{y})$ represent a <u>viable alternative</u> to y and c(y). In particular, an alternative where those who benefited at y and c(y) still do so at $\tilde{y}$ and $\hat{c}(\tilde{y})$, i.e., for each $i \in N$ such that $u^i((I^i - c^i(y)),y^i) > u^i(\overline{x}^i,0)$, what holds true by (31) is that $u^i((I^i - \hat{c}^i(\tilde{y})),\tilde{y}^i) > u^i(\overline{x}^i,0)$. Furthermore, $\tilde{y}$ and $\hat{c}(\tilde{y})$ is an alternative where only those who benefit from $\tilde{y}$ pay for its costs because $\hat{c}(\tilde{y})$ satisfies (26) and (27). Now, if there exist viable alternatives to a given arrangement y and c(y), as above, one can argue that c(y) favors some of those who benefit at the expense of those who do not, even if, by (30), the latter are indifferent between y and c(y) on one hand and the non-provision on the other. In particular, those who benefit relative to c(y) may not be able to cover the costs of y on their own.[11] Thus, we can exclude cost allocation schemes c(y), as above, from being subsidy-free, and we can undertake the following step.

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[11]One, of course, can argue on similar grounds that any cost allocation scheme c(y) that cannot satisfy both (26) and (27) should be excluded from being subsidy-free, independently of whether there exist viable alternatives or not. However, we stop short of doing so because, as we have indicated, in our process of elimination so far we follow the principle that for a cost allocation scheme c(y) to be excluded from being subsidy-free we can find some alternative where the corresponding cost allocation scheme can qualify as being subsidy-free.

Step 3: If there is no element in $F(y)$ that can satisfy all conditions (26), (27), and (30), no cost allocation scheme $c(y) \varepsilon F(y)$ should be considered subsidy-free if there exists a viable alternative to y and $c(y)$.

For each feasible vector of the service y that satisfies the Rationality in Provision condition, let us denote by $SF_M(y)$ the set of cost allocation schemes in $F(y)$ that cannot be eliminated through Steps 1-3, above, as involving some form of cross-subsidization. Apparently, cost allocation schemes that satisfy the symmetry condition (Condition 2) have not been eliminated through the above process. Therefore, every cost allocation scheme $c(y) \varepsilon SF_M(y)$ will not violate any of our four conditions of (a) efficiency, (b) self-finance, (c) pay if you benefit, and (d) symmetry. As a consequence, under one of the weakest notions of when we can call a cost allocation scheme subsidy-free, relative to a vector of the service y, we can do so for every element in the set $SF_M(y)$. In simple terms, this notion is based on the principle that a cost allocation scheme is subsidy-free only if, whenever possible, those who benefit from the service cover its costs. In this sense then, we can say that $SF_M(y)$ represents the maximum number of cost allocation schemes that can be considered subsidy-free relative to a vector of the service y that satisfies the Rationality in Provision condition (for this reason we have used the subscript M to denote this set).

Although the above notion of cross-subsidization does not violate any of the conditions that we have set forth for this purpose, it does not answer the question of how those who benefit from the service should share the costs. However, assuming that a set $SF_M(y)$ contains more than a single element one could continue the process of elimination by undertaking additional steps to those taken above. Nonetheless, such steps may require some notion of fairness and interpersonal comparison of utilities.

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