MULTI-ISSUE COOPERATIVE GAMES

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Abstract

n-Person decision situations often arise where the participants must deal with a variety of issues simultaneously. When binding agreements may be made (cooperative decision situations), traditionally, little attention has been paid to the fact that in such cases the participants may follow strategies that result in the simultaneous formation of different coalition structures with respect to different issues. Introducing cooperative games that allow the players to take full advantage of such strategies, suggesting a method for defining solution concepts for such games, and discussing some applications, is the primary purpose of this paper.

February 1986

1. Introduction

Multi-issue cooperative decision situations differ considerably from single-issue ones in both the strategic possibilities available to the set of participants and the nature of binding agreements that can be reached in each case. For example, in the former case the set of participants have available joint strategies that yield outcomes where no coalition in any single coalition structure can claim that the corresponding to its members vector of levels of utility has been attained through their mutual cooperation. Each one of those members may have cooperated with an entirely different group regarding each of the issues. As a consequence, cooperative games that treat multi-issue cooperative decision situations as single-issue games unduly restrict the strategies available to the participants and, in general, may lead to unsatisfactory theories of coalition formation.

This is best illustrated by trying to represent cooperative decision situations in an economy where the different economic activities (production of various commodities, trade, etc.) can be seen as representing different issues to be dealt with simultaneously by the set of economic agents. The standard approach has been the introduction of assumptions where, in essence, each coalition of economic agents, by pooling the resources of its members, behaves as an economy on its own.

At best, any theory of coalition formation that can emerge from this setting will predict the alignment of economic agents into a single coalition structure. Furthermore, because this setting will not necessarily yield Pareto optimal outcomes, additional assumptions have to be introduced that, in general, make the grand coalition the only outcome of the process.

Although this approach may be suitable for pure exchange economics, and some other simple cases, the predictions of the theory is counter-intuitive and contrary to what von Neumann and Morgenstern (1953, § 4.7) would seem that they had in mind. (Seldomly do we observe single groups of people cooperating in every aspect of economic activity. In general, people do different things with different groups of people.) If, in the terminology of Arrow (1970), the <u>organization of economic activity</u> in an economy (firm formation, operation of markets, etc.) is to be seen as part of the decision process followed by the set of economic agents, furthermore, as long as such organization is to be interpreted as the outcome of cooperation among these agents, cooperative games that allow such outcomes are needed.

Interestingly enough some of the problems that arise in multi-issue cooperative decision situations are completely absent in non-cooperative ones. Beyond the obvious fact that in the latter the formation of coalitions is not an issue, non-cooperative games are in forms (e.g., normal) that provide "for an explicit statement of strategic possibilities," Rosenthal (1972, p. 89). This raises the question whether some of the problems mentioned above, as well as other problems mentioned in Rosenthal (1972) relative to the characteristic function form, can be eliminated by using directly the normal form of a game as a model for (multi-issue, or single-issue) cooperative decision situations; in particular, if in such models the formation of coalitions were to be made internal to the decision problem of each participant.

One of the criticisms often raised about the normal form of a cooperative game is that it is not concerned with the "power" of each coalition, and, therefore, seems inadequate for defining solution concepts. However, this is misleading because there is an underlying normal form for any cooperative game in a form that is concerned with the "power" of each coalition from which the latter can be derived by specifying, in each case, what is meant by "power of a coalition." In this sense, cooperative games in <u>effectiveness form</u> (Rosenthal (1972)) and in <u>characteristic function form</u> (either α- or β-derivation), both derived from a given game in normal form, are different only in the way that that term (power of a coalition) is defined in each case. As a consequence, and as we shall see in the conclusion of this paper, the <u>characteristic function</u> (either α- or β-derivation) is an <u>effectiveness function</u> in the sense of Rosenthal (1972). (In other words, the characteristic function form of a game is a special case of an effectiveness form.)

What these observations suggest are the following.

a. In order to avoid unnecessary restrictions on the strategic possibilities available to the set of participants in a cooperative decision situation we can always use a cooperative game in normal form. Furthermore, in order to obtain satisfactory theories of coalition formation for those cases where the participants must deal with a variety of issues simultaneously, we can make the formation of coalitions relative to each issue internal to the decision problem of each participant. This approach is followed in Section 2.

b. Solution concepts can be defined directly for cooperative games in normal form without having first to derive another form. A notion of "effectiveness," expressing the power of each coalition, and a notion of "justified threats," specifying when a coalition is "justified" in using that power, can be combined for that purpose. The advantage of this approach, that we follow in Section 3 in order to define solution concepts for multi-issue cooperative games, is that it permits direct comparison of different solution concepts in a given class. For example, by fixing the notion of what constitutes a "justified threat" one can vary the notion of "effectiveness" and compare the resulting class of solution concepts. Conversely, one can keep fixed the notion of "effectiveness" and consider variation in the notion of "justified threats."

Some applications of multi-issue cooperative games to the theory of the "core of an economy," the theory of clubs, and voting in committees, are discussed in Section 4.

2. Multi-Issue Games

2.1 Preliminaries

Let N be the set of participants in a multi-issue n-person decision situation. C denotes the collection of all non-empty subsets of N (coalitions), and $C^i = \{c \in C: \ i \in c\}$, for each $i \in N$. The Greek letter τ denotes a partition of N (a coalition structure) and T denotes the set of all such partitions. The set of issues that must be dealt with, simultaneously, is denoted by J; $J = \{1,\ldots,k\}$.

In dealing with each issue $j \in J$, it is assumed that the formation of coalitions is internal to the decision problem of each participant. Thus, for each $j \in J$, each participant $i \in N$ must choose a strategy that consists of two parts: (a) a commitment component l^i_j, and (b) a coalition component c^i_j. In ordinary single-issue games, where individual strategies do not include, explicitly, a coalition component, what we call here a commitment component constitutes the "usual strategy" of a player. We write s^i_j, $s^i_j = (l^i_j;c^i_j)$, $i \in N$, $j \in J$, to denote an individual strategy.

Let L^i_j denote the set of commitments available to the i-th participant relative to the j-th issue. Then, $s^i_j \in S^i_j$, where $S^i_j = L^i_j \times C^i$, for each $i \in N$, and for each $j \in J$. Let $S^i = S^i_1 \times \ldots \times S^i_k$, $i \in N$, and let $S = S^1 \times \ldots \times S^n$. An element $s^i \in S^i$, $s^i = (s^i_1,\ldots,s^i_k)$, $i \in N$, constitutes an individual joint strategy, while an element $s \in S$, $s = (s^1,\ldots,s^n)$, constitutes a joint strategy for the set of participants N.

For each joint strategy $s \in S$, an outcome correspondence ψ determines a set of final outcomes. Thus, if Y denotes the set of all final outcomes, $\psi: S \twoheadrightarrow Y$, and $\psi(s) \subseteq Y$.

Final outcomes in Y are ordered by each participant $i \in N$ through that participant's preference relation $\gtrsim_i$. We write $R = (\gtrsim_1,\ldots,\gtrsim_n)$ to denote a profile, and $(\tilde{y} >_i y) \Leftrightarrow (\tilde{y} \gtrsim_i y$ but not $y \gtrsim_i \tilde{y})$, for any two final outcomes $\tilde{y}$, $y \in Y$, and for any $i \in N$.

A 6-tuple $(N,J,S,Y,\psi,R) = G$ that satisfies the above specifications constitutes an n-person multi-issue game in normal form.

2.2 Cooperative Games

A game G "becomes cooperative if we allow the players to communicate before each play and to make binding agreements about the strategies they use," Aumann (1967, p. 3). As a consequence, only certain subsets of S are compatible with a multi-issue cooperative decision situation.

To obtain such subsets, first, we must take into account any constraints that may be imposed on the commitment components of each individual joint strategy due to the fact that each player must deal with the entire set of issues J simultaneously. Let $\hat{L}^i$, $\hat{L}^i \subseteq L^i$, denote the set that satisfies all such constraints, and let $\hat{S}^i = \{s^i \in S^i: 1^i \in \hat{L}^i\}$, for each $i \in N$. We refer to $\hat{S}$, $\hat{S} = \hat{S}^1 \times \ldots \times \hat{S}^n$, as the <u>set of individually feasible joint strategies</u> of G.

Our second concern lies with the coalition components of each strategy. We say that a <u>joint strategy</u> $s \in S$ <u>admits coalition</u> c, $c \in C$, <u>with respect to issue</u> j, $j \in J$, if and only if $s^i_j = (1^i_j;c)$, $\forall\ i \in c$. A coalition so admitted by a joint strategy s is called an s-<u>admissible coalition</u>. $C(s) = \{c \in C: \exists\ j,\ j \in J,\ \ni:\ s^i_j = (1^i_j;c),\ \forall\ i \in c\}$, denotes the <u>set of</u> s-<u>admissible coalitions</u>, for each $s \in S$.

A joint strategy $s \in S$ is <u>consistent</u> if and only if $c^i_j \in C(s)$, $\forall\ i \in N$, and $\forall\ j \in J$, where, given s, c^i_j represents the coalition component of s^i_j. S^* denotes the <u>set of consistent joint strategies</u> for a game G.

For each $s \in S$, and for each $j \in J$, let $C_j(s)$ denote the set of coalitions that are s-admissible with respect to issue j, $j \in J$. It follows that

(1) $(s \in S^*) \Leftrightarrow (\text{For each } j \in J,\ \exists \tau_j,\ \tau_j \in T,\ \ni\ C_j(s) = \bigcup_{c \in \tau_j} \{c\})$.

Let $\hat{S}^*$, $\hat{S}^* = \hat{S} \cap S^*$, denote the <u>set of individually feasible consistent joint strategies</u> of G, and let $\hat{\psi}$ denote the restriction of ψ over the set $\hat{S}^*$. Then, the 6-tuple Γ,

(2) $$\Gamma = (N,J,\hat{S}^*,Y,\hat{\psi},R),$$

can be considered as an n-person multi-issue cooperative game. Furthermore, Γ is the cooperative game with the largest set of joint strategies that we can obtain from G (i.e., joint strategies $s \in (S-S^*)$ cannot be the result of communication among the players, while joint strategies $s \in (S-\hat{S})$ cannot be binding for every player).

For comparative purposes, let ρ: $\hat{S}^* \to \hat{S}^*$ denote a reordering of joint strategies in $\hat{S}^*$ defined by

(3) $$\rho(s) = (((1_j^i;c)_{i \in c})_{c \in \tau_j^s})_{j \in J}$$

where, by (1), for each $s \in \hat{S}^*$, τ_j^s represents the coalition structure corresponding to the j-th issue. Furthermore, let

(4) $$\tilde{S} = \{s \in \hat{S}^*: \rho(s) = (((1_j^i;c)_{i \in c})_{c \in \tau_j^s})_{j \in J}, \text{ and } \tau_j^s = \tau^s, \tau^s \in T, \forall j \in J\},$$

and let

(5) $$\tilde{\Gamma} = (N,J,\tilde{S},Y,\tilde{\psi},R)$$

where $\tilde{\psi}$ represents the restricrtion of ψ over the set $\tilde{S}$.

Then, $\tilde{\Gamma}$ is also a cooperative game, albeit, one equivalent to treating multi-issue decision situations as single-issue games. (The joint strategies in $\tilde{S}$ allow the players to form only common coalition structures with respect to each and every issue $j \in J$, which is equivalent to forcing each coalition to deal internally with all issues.) We will use $\tilde{\Gamma}$ throughout this paper in parallel with Γ in order to make comparisons between the two approaches.

2.3 Strategies for Coalitions

Given a game Γ, for each $c \in C$, and for each $j \in J$, let $a_j^c = (1_j^i;c)_{i \in c}$, so that, for each $s \in \hat{S}^*$, we can rewrite (3) as $\rho(s) = ((a_j^c)_{c \in \tau_j^s})_{j \in J}$. To the extent that in an n-person cooperative decision situation a_j^c can be seen as representing a potential binding agreement among the members of c relative to issue j, we refer to it as a coalition action. In this sense, then, the reording $\rho(s)$ of each joint strategy $s \in \hat{S}^*$ consists of coalition actions, one for each s-admissible coalition $c \in C(s)$. For simplicity, let us write $a_j^c \in \rho(s)$

to indicate that the coalition action a_j^c is an element of $\rho(s)$, and adopt similar notation throughout this paper.

For each $c \varepsilon C$, and for each $j \varepsilon J$, let $\hat{A}_j^{*c} = \{a_j^c\colon\ \exists\ s,\ s \varepsilon \hat{S}^*,$ and $a_j^c \varepsilon \rho(s)\}$, and for each $c \varepsilon C$, let $\hat{A}^{*c} = \bigcup_{j \varepsilon J} \hat{A}_j^{*c}$. Furthermore, for each $s \varepsilon \hat{S}^*$, for each coalition structure τ_r^s, $r \varepsilon J$ (guaranteed by (1) with respect to s), and for each coalition $\tilde{c} \varepsilon C$, let $\{\tau_r^s(\tilde{c}),$ $\tau_r^s(N-\tilde{c})\}$ denote the partition of τ_r^s defined by: $\tau_r^s(\tilde{c}) = \{c \varepsilon \tau_r^s\colon\ c \cap \tilde{c} \neq \emptyset\}$, and $\tau_r^s(N-\tilde{c}) = \tau_r^s - \tau_r^s(\tilde{c})$. Then, for each $s \varepsilon \hat{S}^*$, and for each $\tilde{c}$, $\tilde{c} \varepsilon C$, we can regroup the coalition actions in $\rho(s)$ into those that involve some member of $\tilde{c}$ and those that do not involve any member of $\tilde{c}$. That is, if

$$a(\tilde{c}) = ((a_r^c)_{c \varepsilon \tau_r^s(\tilde{c})})_{r \varepsilon J}, \text{ and}$$

$$a(N-\tilde{c}) = ((a_r^c)_{c \varepsilon \tau_r^s(N-\tilde{c})})_{r \varepsilon J},$$

we can denote this regrouping of coalition actions by $\rho^{\tilde{c}}(s)$, where $\rho^{\tilde{c}}(s) = (a(\tilde{c});a(N-\tilde{c}))$.

Note that $(\tilde{c} \varepsilon C(s)) \Rightarrow (\exists j, j \varepsilon J,\ \ni \tau_j^s(\tilde{c}) = \{\tilde{c}\}$, and $\exists\ a_j^{\tilde{c}}$, $a_j^{\tilde{c}} \varepsilon \hat{A}^{*\tilde{c}},\ \ni a_j^{\tilde{c}} \varepsilon a(\tilde{c}))$, for each $s \varepsilon \hat{S}^*$. Therefore, the correspondence $\eta(\cdot)$, where

(6) $\eta(a_j^c) = \{a(c)\colon\ \exists\ s,\ s \varepsilon \hat{S}^*,\ \ni a_j^c \varepsilon \rho(s)$, and $\rho^c(s) =$
$(a(c);a(N-c))\}$,

associates each coalition action $a_j^c \varepsilon \bigcup_{c \varepsilon C} \hat{A}^{*c}$ with all vectors of coalition actions $a(c)$ that contain it, that is, $(a(c) \varepsilon \eta(a_j^c)) \Leftrightarrow (a_j^c \varepsilon a(c))$.

A <u>joint strategy for a coalition</u> c, $c \varepsilon C$, in a game Γ, is a vector of individual joint strategies $((s_r^i)_{r \varepsilon J})_{i \varepsilon c}$ that satisfies the following two conditions:

a. (<u>admissibility</u>): $\exists\, a_j^c,\ a_j^c \in \hat{A}^{*c},\ \ni\ (s_j^i)_{i \in c} = a_j^c$; and

b. (<u>consistency</u>): $\exists\, s,\ s \in \hat{S}^*,\ \ni\ s = (((s_r^i)_{r \in J})_{i \in c},$ $((s_r^i)_{r \in J})_{i \in N-c})$.

In essence, the above definition of joint strategies for coalitions in a game Γ is equivalent to the following. First, the members of a coalition $c \in C$ get together and they agree to cooperate relative to some issue $j \in J$, thus, undertaking a coalition action $a_j^c \in \hat{A}^{*c}$ (admissibility). Second, each member pursues to make additional aggrements relative to the remaining issues that, overall, lead to a vector of coalition actions $a(c) \in \eta(a_j^c)$ (consistency). The vector of individual joint strategies $((s_r^i)_{r \in J})_{i \in c}$ associated with the pair $(a_j^c;a(c))$, then, is what constitutes a joint strategy for that coalition.

Because the only coalition action guaranteed through a joint strategy $((s_r^i)_{r \in J})_{i \in c}$, for some $c \in C$, is the corresponding $a_j^c \in \hat{A}^{*c}$, while every other coalition action in $a(c)$ may be subject to what strategies players in N-c decide to follow, joint strategies for coalitions in a game Γ have a conditional feature in them. Furthermore, there will be no loss of generality if we use pairs $(a_j^c;a(c))$, $c \in C$, as strategies for coalitions, and write

$$(a_j^c | a(c))$$

to indicate their conditional feature.

For each $c \in C$, let

(7) $\hat{S}^*(c) = \{(a_j^c | a(c)):\ \ a_j^c \in \hat{A}^{*c}, \text{ and } a(c) \in \eta(a_j^c)\}$.

We will refer to $\hat{S}^*(c)$ as the <u>set of conditional strategies</u> for coalition c.

For comparative purposes, for each $c \in C$, let

(8) $\tilde{S}(c) = \{(a_j^c | a(c)) \in \hat{S}^*(c): \exists\, s,\ s \in \tilde{S}, \text{ and } \rho^c(s) = (a(c);a(N-c))\}$.

The corresponding to each $(a_j^c | a(c)) \in \tilde{S}(c)$, $c \in C$, joint strategy does not have the conditional feature mentioned above. In this case, a coalition undertakes a coalition action with respect to each and every issue $j \in J$. Thus, its members do not pursue to make binding agreements with players in N-c, and the coalition actions in a(c) are obtainable by just reordering the elements of the respective joint strategy $((s_r^i)_{r \in c}$. In contrast to $\hat{S}^*(c)$, then, we can refer to $\tilde{S}(c)$ as the set of <u>independent strategies</u> for coalition c, $c \in C$.

Because the sets $\tilde{S}(c)$ will correspond to the joint strategies of each coalition $c \in C$ in a game $\tilde{\Gamma}$, it is easily seen that the conditional feature of joint strategies mentioned above is different from that considered by Rosenthal (1972) whose cooperative games in <u>effectiveness form</u> can be derived from a game $\tilde{\Gamma}$.

3. Solution Concepts

Given an n-person multi-issue cooperative game Γ, let Z,

$$Z = \{(s,y): \ s \in \hat{S}^*,\ y \in \hat{\Psi}(s)\},$$

denote the set of <u>alternatives</u> available to the set of players N.

The method suggested below for defining solution concepts for an n-person cooperative game deals directly with the form of Γ (or $\tilde{\Gamma}$) given in the preceding section. It utilizes an "effectiveness rule" and a "threat correspondence" for that purpose. Furthermore, in order to apply the notion of stability implied by each solution concept to the coalition structures that yield the relevant outcomes, we define solution concepts over the set of alternatives available to the players in each game Γ (or $\tilde{\Gamma}$).

3.1 Effectiveness Rules

A coalition that uses a conditional strategy $(\tilde{a}_j^c|\tilde{a}(c))$ in hope that, after all agreements have been completed, the alternative $(\tilde{s},\tilde{y})$ will emerge, where $\rho^c(\tilde{s}) = (\tilde{a}(c);\tilde{a}(N\text{-}c))$, must concern itself with two levels of strategic behavior by the players in (N-c). The first level concerns coalition actions in $\tilde{a}(c)$ that require the participation of players in (N-c), while the second level concerns coalition actions in $\tilde{a}(N\text{-}c)$.

Consent Correspondences

To deal with the first level of strategic behavior, above, let Θ denote a set of indices. For each $\theta \in \Theta$, a correspondence

$$\zeta_\theta : C \times Z \twoheadrightarrow \bigcup_{c \in C} \hat{S}^*(c) \tag{9}$$

specifies, for each coalition $c \in C$, and for each alternative $(s,y) \in Z$, the conditional strategies $(\tilde{a}_j^c|\tilde{a}(c)) \in \hat{S}^*(c)$ that if followed by coalition c its members can be assured that, whenever the participation of players in N-c is required for coalition actions in $\tilde{a}(c)$, these players cannot refuse such participation and, thus, they give their consent to such coalition actions.

The set Θ, then, indexes all possible ways that a coalition can be assured of the consent of players in N-c as above. We refer to each correspondence ζ_θ, $\theta \in \Theta$, in (9) as a consent correspondence for a game Γ.

Because $\tilde{S}(c) \subseteq \zeta_\theta(c;(s,y)) \subseteq \hat{S}^*(c)$, for each $c \in C$, for each $(s,y) \in Z$, and for any $\theta \in \Theta$, we will use the indices θ^o and θ^* to represent the two extreme cases of a consent correspondence. That is, we shall let,

(10) $\zeta_{\theta^o}(c;(s,y)) = \tilde{S}(c)$, for each $c \in C$, and $\forall\ (s,y) \in Z$, and

(11) $\zeta_{\theta*}(c;(s,y)) = \hat{S}*(c)$, for each $c \in C$, and $\forall\ (s,y) \in Z$.

Therefore, $\zeta_{\theta^o}(c:(s,y)) \sqsubseteq \zeta_\theta(c;(s,y)) \sqsubseteq \zeta_{\theta*}(c;(s,y))$, for each $c \in C$, for each $(s,y) \in Z$, and $\forall\ \theta \in \Theta$.

In the case of θ^o, independently of the alternative under consideration, players in N-c never give their consent, when needed, to coalition actions in $\tilde{a}(c)$, and coalition c must use only independent strategies. In the case of $\theta*$, players in N-c always give their consent, when needed, to coalition actions in $\tilde{a}(c)$, and coalition c can use any of its conditional strategies in $\hat{S}*(c)$.

Enforcement Rules

To deal with the second level of strategic behavior mentioned above, let

$$\phi:\quad \bigcup_{c \in C} \hat{S}*(c) \twoheadrightarrow Z$$

be the correspondence defined by

(12) $\phi(\tilde{a}^c_j|\tilde{a}(c)) = \{(\hat{s},\hat{y}) \in Z:\ \rho^c(\hat{s}) = (\hat{a}(c);\hat{a}(N\text{-}c))$ and $\hat{a}(c) = \tilde{a}(c)\}$,

and let P(Z) denote the power set of Z.

An <u>enforcement rule</u> for a game Γ is a correspondence

$$\varepsilon:\quad \bigcup_{c \in C} \zeta_\theta(c;(s,y)) \twoheadrightarrow P(Z) \tag{13}$$

that specifies for each $(\tilde{a}^c_j|\tilde{a}(c)) \in \bigcup_{c \in C} \zeta_\theta(c;(s,y))$ the set of alternative subsets of $\phi(\tilde{a}^c_j|\tilde{a}(c))$ which the respective coalition c can <u>enforce</u> against (s,y), given $\theta \in \Theta$. To paraphrase Rosenthal (1972, p. 93), when faced with the alternative (s,y), coalition c by using the conditional strategy $(\tilde{a}^c_j|\tilde{a}(c)) \in \zeta_\theta(c;(s,y))$ is able in the sense specified by ε to restrict the negotiation process (at least temporarily) to any one of the specified sets in $\varepsilon(\tilde{a}^c_j|\tilde{a}(c))$. However, if such a specified set contains more than a single alternative, coalition c cannot determine which particular alternative will result without the concurrence of members of N-c regarding coalition actions in $\hat{a}(N-c)$.

We denote by E the set of all enforcement rules that can be defined relative to a game Γ.

Notions of Effectiveness

Let $\Lambda = \Theta \times E$, with arbitrary elements denoted by the Greek letter λ, i.e., $\lambda = (\theta, \varepsilon)$. For each $\lambda \in \Lambda$, let $\xi_\lambda : C \times Z \twoheadrightarrow P(Z)$ be the correspondence defined by

$$(14)\quad \xi_\lambda(c;(s,y)) = \bigcup_{(\tilde{a}_j^c | \tilde{a}(c)) \in \zeta_\theta(c;(s,y))} \varepsilon(\tilde{a}_j^c | \tilde{a}(c)), \text{ for each } c \in C,$$

and for each $(s,y) \in Z$.

The set $\xi_\lambda(c;(s,y))$, then, represents the only subsets of Z that coalition c can enforce (in the sense specified by ε) against the alternative (s,y) as it varies its strategies over the set (specified by θ) $\zeta_\theta(c;(s,y))$. In this sense, $\xi_\lambda(c;(s,y))$ represents the collection of subsets of Z for which coalition c is λ-effective relative to (s,y).

Therefore, for each $\lambda \in \Lambda$, we can say that ξ_λ represents an effectiveness rule for a game Γ, while Λ indexes all such rules.

Let $\Lambda^0 = \{\theta^o\} \times E$. By (10), Λ^0 indexes all effectiveness rules of a game Γ that we could have obtained from a game $\tilde{\Gamma}$, as well. In particular, because for a game $\tilde{\Gamma}$ each index $\lambda \in \Lambda$ can be obtained solely on the basis of the corresponding enforcement rule ε, Λ^0 represents the range of all such indices for $\tilde{\Gamma}$.

3.2 Threat Correspondences

An element $X \in P(Z)$ constitutes a threat by a coalition $c \in C$ against an alternative $(s,y) \in Z$, if and only if,

(15) for each $(\tilde{s},\tilde{y}) \in X$, $\exists\ \tilde{a}_j^c, \tilde{a}_j^c \in \rho(\tilde{s})$, $\ni$: $\tilde{a}_j^c \notin \rho(s)$; and

(16) $\exists\, X',\ X' \subseteq X,\ \ni:$ for each $(\tilde{s},\tilde{y}) \in X'$, $\tilde{y} \gtrsim_i y$, $\forall\, i \in c$, and

$\tilde{y} >_r y$, for some $r \in c$.

A threat correspondence for a game Γ is a correspondence

(17) $$\mu:\ C \times Z \twoheadrightarrow P(Z)$$

that, for each $(s,y) \in Z$, specifies the set of alternative subsets of Z that each coalition $c \in C$ is "justified" in using as threats against (s,y).

Different notions of what constitutes a "justified threat" will lead to different threat correspondences. In general, such notions will require that (13) holds, in each case, for subsets X' of X that satisfy certain requirements. For example, in addition to considering every threat as justified, other threat correspondences can be obtained by requiring that (13) holds for X' = X, or for those subsets X' of X for which there do not exist "counter-threats."

The letter M denotes the set of all threat correspondences that can be defined relative to a game Γ.

3.3 Notions of Stability

For each $\lambda \in \Lambda$, and for each $\mu \in M$, let

(18) $$\Sigma_\mu^\lambda(\Gamma) = \{(s,y) \in Z:\ \ \xi_\lambda(c;(s,y)) \cap \mu(c;(s,y)) = \emptyset,\ \forall\, c \in C\}.$$

In the terminology introduced above, for each $(s,y) \in \Sigma_\mu^\lambda(\Gamma)$, the alternative subsets of Z for which each coalition $c \in C$ is λ-effective relative to (s,y) cannot be used as justified threats (in the sense specified by μ) against (s,y). In this sense, then (s,y) is stable, and $\Sigma_\mu^\lambda(\Gamma)$ constitutes a solution concept for Γ (albeit one that includes both final outcomes and the corresponding to them joint strategies in $\hat{S}^*$).

For comparative purposes, let

$$\tilde{Z} = \{(s,y) \in Z: \; s \in \tilde{S}\}.$$

Application of the notion of stability implied by each pair (λ,μ) to a game $\tilde{\Gamma}$ (i.e., by considering the restrictions of the respective correspondences over the relevant to $\tilde{\Gamma}$ sets) will yield

(19) $\Sigma_{\mu}^{\lambda}(\tilde{\Gamma}) = \{(s,y) \in \tilde{Z}: \; \xi_{\lambda}(c;(s,y)) \cap \mu(c;(s,y)) = \emptyset, \; \forall \; c \in C\}$,

as the corresponding to $\tilde{\Gamma}$ solution concept. However, effectivenss rules for a game $\tilde{\Gamma}$ are invariant with respect to θ. Therefore,

(20) $\Sigma_{\mu}^{\bar{\lambda}}(\tilde{\Gamma}) = \Sigma_{\mu}^{\bar{\lambda}^{o}}(\tilde{\Gamma})$, $\forall \; \theta \in \Theta$, for each $\bar{\varepsilon} \in E$, and for each $\mu \in M$,

where, for each $\bar{\varepsilon} \in E$, $\bar{\lambda}^{o} = (\theta^{o},\bar{\varepsilon})$, i.e., $\lambda^{o} \in \Lambda^{0}$, and $\bar{\lambda} = (\theta,\bar{\varepsilon})$.

As a consequence, a game $\tilde{\Gamma}$ restricts, not only the set of alternatives available to the set of players, in a given cooperative decision situation, from Z to $\tilde{Z}$, but the threat possibilities that may be available to those players as well. In other words, in a game $\tilde{\Gamma}$ a coalition c is never able to use strategies in the set $\hat{S}^{*}(c) - \tilde{S}(c)$ to initiate threats.

3.4 Theories of Coalition Formation

The inclusion of a coalition component c_j^i in each individual strategy s_j^i, $i \in N$, $j \in J$, and the definition of solution concepts for a game Γ over the set of alternatives Z, imply that inherent with each solution concept $\Sigma_{\mu}^{\lambda}(\Gamma)$ is a theory of coalition formation. In particular, for each $(s,y) \in \Sigma_{\mu}^{\lambda}(\Gamma)$ there exist, by (1), k coalition structures τ_j^s, $j \in J$, associated with the joint strategy s, that satisfy the same notion of stability that (s,y) does. But then,

(21) $C\Sigma_{\mu}^{\lambda}(\Gamma) = \{(\tau_j^s)_{j \in J}: \; (s,y) \in \Sigma_{\mu}^{\lambda}(\Gamma)\}$

will be the set of all vectors of coalition structures $(\tau_j^s)_{j \in J}$ that we can consider stable, given $\lambda \in \Lambda$ and $\mu \in M$, for a game Γ.

Naturally, for a game $\tilde{\Gamma}$, $(\tau_j^s)_{j \varepsilon J} \varepsilon C\Sigma_\mu^\lambda(\tilde{\Gamma})$ only if $\tau_j^s = \tau_r^s$, $\forall$ r, $j \varepsilon J$, which shows the limitations of such games.

3.5 Cores and Other Solution Concepts

Let us demonstrate how the process leading to (18) and (19) can be applied to obtain specific families of solution concepts.

α-Cores

For each $\theta \varepsilon \Theta$, and for each $(s,y) \varepsilon Z$, let ε^α, $\varepsilon^\alpha \varepsilon E$, be the enforcement rule defined by

(22) $\varepsilon^\alpha(\tilde{a}_j^c|\tilde{a}(c)) = \{\phi(\tilde{a}_j^c|\tilde{a}(c))\}$, for each $(\tilde{a}_j^c|\tilde{a}(c)) \varepsilon \bigcup_{c \varepsilon C} \zeta_\theta(c;(s,y))$.

Furthermore, let μ^α, $\mu^\alpha \varepsilon M$, be the threat correspondence defined by

(23) $(X \varepsilon \mu^\alpha(c;(s,y))) \Leftrightarrow ((13)$ holds for $X' = X)$, for each $c \varepsilon C$, and for each $(s,y) \varepsilon Z$,

where X denotes a threat.

For $\lambda = (\varepsilon^\alpha,\theta)$, the collection of subsets of Z $\xi_\lambda(c;(s,y)) \cap \mu^\alpha(c;(s,y))$, when non-empty for some $(s,y) \varepsilon Z$ and for some $c \varepsilon C$, can contain only sets of the form $\phi(\tilde{a}_j^c|\tilde{a}(c))$, for $(\tilde{a}_j^c|\tilde{a}(c)) \varepsilon \zeta_\theta(c;(s,y))$. By (12), this means that, once the coalition actions in $\tilde{a}(c)$ have been taken care of (in the sense specified by θ), coalition c by following $(\tilde{a}_j^c|\tilde{a}(c)) \varepsilon \zeta_\theta(c;(s,y))$ can be assured of an alternative that "independently of the strategies followed by its complement N-c" will satisfy the preference requirements in (16). In this sense, the solution concepts obtained from (22) and (23) as we vary θ over Θ preserve some of the characteristics of an α-core (see, e.g., Aumann and Peleg (1960)). In fact, by (20), if we were to restrict this analysis to a game $\tilde{\Gamma}$, the only solution concept that we will be able to obtain by using $\varepsilon = \varepsilon^\alpha$ and $\mu = \mu^\alpha$ is the α-core of $\tilde{\Gamma}$, denoted here by $K^\alpha(\tilde{\Gamma})$, independently of any $\theta \varepsilon \Theta$.

For each $\theta \varepsilon \Theta$, let $K_{\theta}^{\alpha}(\Gamma)$ denote the solution concept of a game Γ obtainable by using $\varepsilon = \varepsilon^{\alpha}$, and $\mu = \mu^{\alpha}$, in (18). Note that $K_{\theta^o}^{\alpha}(\Gamma)$ differs from $K^{\alpha}(\tilde{\Gamma})$ only in that the former is defined over the set of alternatives Z while the latter is defined over the set of alternatives $\tilde{Z}$. But then, $K_{\theta^o}^{\alpha}(\Gamma)$ represents the α-core of Γ.

Now, the only difference between $K_{\theta^o}^{\alpha}(\Gamma)$ and $K_{\theta}^{\alpha}(\Gamma)$, for any $\theta \varepsilon \Theta$, $\theta \neq \theta^o$, is in what sense a coalition c can be assured that players in N-c give their consent to coalition actions requiring cooperation of players in both groups. In this sense, then, for each $\theta \varepsilon \Theta$, we can refer to $K_{\theta}^{\alpha}(\Gamma)$ as the α_{θ}-core of Γ, with $K_{\theta^o}^{\alpha}(\Gamma)$ representing the α-core of Γ in the "usual sense."

Other Core-Concepts

At the other extreme of the family of α-cores of Γ is the family of, what we refer to, the cores* of Γ. We obtain this family by using the enforcement rule ε^*, where, for $\theta \varepsilon \Theta$, and for each $(s,y) \varepsilon Z$,

(24) $$\varepsilon^*(\tilde{a}_j^c|\tilde{a}(c)) = \{\{(\tilde{s},\tilde{y})\}: (\tilde{s},\tilde{y}) \varepsilon \phi(\tilde{a}_j^c|\tilde{a}(c))\}, \text{ for each } (\tilde{a}_j^c|\tilde{a}(c)) \varepsilon \bigcup_{c \varepsilon C} \zeta_{\theta}(c;(s,y)),$$

and the threat correspondence μ^*, where,

(25) $$(X \varepsilon \mu^*(c;(s,y))) \Longleftrightarrow (X \text{ is a threat}), \text{ for each } c \varepsilon C, \text{ and } \forall (s,y) \varepsilon Z.$$

Let $K_{\theta}^*(\tilde{\Gamma})$ denote the core_{θ}^* of $\tilde{\Gamma}$, $\theta \varepsilon \Theta$. By (20), $K_{\theta}^*(\tilde{\Gamma}) = K_{\theta^o}^*(\tilde{\Gamma}) = K^*(\tilde{\Gamma})$, $\forall \theta \varepsilon \Theta$, i.e., there is only one core* for $\tilde{\Gamma}$. However, like the case of the α-cores there is a whole family of cores* for Γ, with each member of that family denoted by $K_{\theta}^*(\Gamma)$, $\theta \varepsilon \Theta$.

In defining one particular member of this family, the $\text{core}_{\theta^*}^*$ of Γ $(K_{\theta^*}^*(\Gamma))$, an implication of (11), (24), and (25) is that each coalition $c \varepsilon C$ can use any of its conditional strategies $(\tilde{a}_j^c|\tilde{a}(c)) \varepsilon \hat{S}^*(c)$, it is effective for any alternative consistent with that strategy (i.e., for any $(\tilde{s},\tilde{y}) \in \phi(\tilde{a}_j^c|\tilde{a}(c))$), and every threat is justified. Furthermore, these implications hold independently of what alternative $(s,y) \in Z$ we take under consideration. An intuitive way of justifying the <u>core</u>$^*_{\theta*}$ of Γ, then, is the following.

Each alternative $(s,y) \in Z$ is considered a <u>proposal</u> that can come up for adoption by the players at any stage of the bargaining process. Given that an alternative $(s,y) \in Z$ has been proposed for adoption, a coalition $c \in C$ is free to counter with another alternative, say $(\tilde{s},\tilde{y})$, as long as there is a coalition action $\tilde{a}_j^c \in \rho(\tilde{s})$ that satisfies (15). The new alternative, if proposed, replaces the original one and it becomes the new proposal under consideration. The set $K^*_{\theta*}(\Gamma)$ consists of all alternatives $(s,y) \in Z$ for which no coalition $c \in C$ has the incentive (in the sense of the preference requirements in (16)) to counter.

Because this is an interpretation that under certain circumstances can be applied in the definition of an α-<u>core</u> as well, it would seem that for certain games Γ, $K^*_\theta(\Gamma) = K^\alpha_\theta(\Gamma)$, for some $\theta \in \Theta$, and consequently $K^*(\tilde{\Gamma}) = K^\alpha(\tilde{\Gamma})$. Indeed, it is easy to prove that this will be the case for any game Γ for which the following condition (26) holds, where $\lambda^* = (\varepsilon^*,\theta)$.

(26) $((((\tilde{a}_j^c|\tilde{a}(c) \in \zeta_\theta(c;(s,y)),\ (\tilde{s},\tilde{y}) \in \phi(\tilde{a}_j^c|\tilde{a}(c)),$ and $\{(\tilde{s},\tilde{y})\} \in \xi_{\lambda*}(c;(s,y)) \cap \mu^*(c;(s,y))) \Rightarrow (\{(\bar{s},\bar{y})\} \in \xi_{\lambda*}(c;(s,y)) \cap \mu^*(c;(s,y)),\ \forall\ (\bar{s},\bar{y}) \in \phi(\tilde{a}_j^c|\tilde{a}(c))))$, for each $c \in C$, and for each $(s,y) \in Z$.

Bargaining Sets

Bargaining set concepts (e.g., Aumann and Maschler (1964), Davis and Maschler (1967), and Peleg (1967)) for a game Γ are easy to obtain by modifying the method suggested by Rosenthal (1972) so that a threat correspondence will include threats for which there do not exist counter-threats as justified ones.

More specifically, let us use the set B to index all possible methods for defining "objections" and "counterobjections" in a game Γ. Using the terminology introduced by Rosenthal (1972, p. 97) we can proceed as follows.

A threat $X \varepsilon P(Z)$ against some alternative $(s,y) \varepsilon Z$ by some coalition $c \varepsilon C$ will be referred to (for this purpose) as an <u>objection set</u>. For given $b \varepsilon B$, a <u>prime objection set in the</u> b-<u>sense</u> is a subset $X' \subseteq X$ that satisfies (16) and some other conditions (like (23), or (25)) that are characteristic to the particular $b \varepsilon B$ under consideration. A pair $(X,X')_b$, as above, is called an <u>objection in the</u> b-<u>sense</u> against (s,y) by coalition c.

A <u>counterobjection in the</u> b-<u>sense</u> to an objection $(X,X')_b$ is an objection in the b-sense, $(\bar{X},\bar{X}')_b$, against alternatives $(s',y') \varepsilon X'$ that, again, depending on the particular $b \varepsilon B$ under consideration, satisfies certain additional conditions relative to both (s,y) and (s',y'). Such conditions, in general, will specify (a) what coalitions are entitled to form counterobjections and (b) requirements about how their members order (s,y), (s',y'), and elements of $\bar{X}'$, in terms of preferences.

A pair $(X,X')_b$ constitutes a <u>justified objection in the</u> b-<u>sense</u>, $b \varepsilon B$, against an alternative $(s,y) \varepsilon Z$ by a coalition $c \varepsilon C$, and X is included in the relevant threat correspondence, denoted here by μ^b, i.e., $X \varepsilon \mu^b(c;(s,y))$, if and only if $(X,X')_b$ is an objection in the b-sense (against (s,y) by c) for which there do not exist counterobjections in the same sense. The so obtained threat correspondence μ^b, then, can be combined with an effectiveness rule ξ_λ, $\lambda \varepsilon \Lambda$, to obtain the corresponding bargaining set concept from (18).

4. Some Applications

4.1 An Economy and Its "Core"

The class of economic environments (economies) where, in the sense of Koopmans (1957, p. 71), the production possibilities are given "before institutional assumptions are specified," is of particular interest here because it is easy to show that each economy in that class is conceptually equivalent to a game G. In particular, for such cases we can always give the following interpretation to the elements of the 6-tuple (N,J,S,Y,ψ,R) that describes G.

N represents the set of economic agents in an economy while J represents the set of economic activities that can be carried on by those agents. Activities such as, exchange, production of various commodities, and, in general, any activity that if followed by a subset of agents is capable of changing the initial allocation of resources, can be included in J.

For each activity $j \varepsilon J$, then, each economic agent $i \varepsilon N$ is involved in a twofold decision process: (a) what amount of resources to devote to that activity (i.e., a choice of l_j^i), and (b) whether to act alone or in cooperation with others (i.e., a choice of $c_j^i \varepsilon C^i$). As a consequence, S can be used to represent all possible decisions available to the set of economic agents relative to J, while $\hat{S}$ will represent decisions where each economic agent $i \varepsilon N$ must make overall commitments of resources that lie in his feasible set $\hat{L}^i$, as the latter is determined by that agent's vector of initial endowments, say, $\bar{l}^i$.

In this case ψ can be interpreted as representing the technology of an economy (an (aggregate) production correspondence) that transforms joint decisions $s \in S$ into final allocations in the set Y. The lack of institutional assumptions, then, is reflected by the nature of joint decisions in S.

A pure exchange economy can be seen as a case where J includes only a single activity, and more complex cases can be obtained by increasing the number of activities included in J. Furthermore, both <u>decomposable</u> (absence of externalities) and <u>non-decomposable</u> economic environments (see, e.g., Hurwicz (1960, p. 33), and Ledyard (1977, p. 1609)) can be taken under consideration.

Application of our analysis in Sections 2 and 3 to deal with cooperative decision situations in a given economic environment like the above is straightforward. An implication of that analysis is that, for a given economy, variations in the organization of economic activity compatible with a given solution concept will be revealed through the set of coalition structures in (21).

Because the "core" has played a prominent role in analyzing cooperative decision situations in an economy, some of the implications of Sections 2 and 3 regarding its definition are worth mentioning here.

Suppose that we start out with a pure exchange economy (only a single activity). Then $\Gamma = \tilde{\Gamma}$, and $K^{\alpha}_{\theta}(\Gamma) = K^{\alpha}_{\theta}(\tilde{\Gamma})$, $\forall\ \theta \in \Theta$. Furthermore, it is easy to show that in the absence of externalities (26) holds and, therefore, for such economic environments $K^{\alpha}(\tilde{\Gamma}) = K^{*}(\tilde{\Gamma})$. However, these equalities may no longer hold for economic environments with a greater number of economic activities or various externalities.

For example, for an economy that includes the production of a pure public good it is not necessary that (26) will hold, and, in such cases we may have $K^*(\tilde{\Gamma}) = \emptyset$ while (see, e.g., Muench (1972)) $K^{\alpha}(\tilde{\Gamma})$ may be "very large." In addition, as we shall demonstrate explicitly in the following subsection, a multiplicity of economic activities in a given economic environment can result in different sets of final outcomes achievable under each game Γ and $\tilde{\Gamma}$.

What is illustrated by this discussion is the following. A number of different core-concepts (in this case different with respect to whether we use Γ or $\tilde{\Gamma}$ to represent cooperative decision situations in an economy, as well as, to the notion of stability used) can coinside for certain classes of simple economic environments. Thus, giving the impression that there is a unique way of defining the "core of an economy." However, because, actually, we may be dealing with different core-concepts, there is not a unique way for extending the notion of the "core of an economy" to every economic environment no matter how complex. As it has been noted by D. Starrett (1973), the "blocking rules" used by Shapley and Shubik (1969) in defining "the core of an economy with external diseconomies" and those used by Foley (1970) in defining "the core of an economy with public goods" are not the same. Nonetheless, either set of rules will yield the same set of stable allocations in an economy without these externalities.

4.2 Theory of Clubs (An Example)

A community consists of three members, i.e., $N = \{1,2,3\}$, each endowed with 1000 units of some commodity x. Commodity x can be used for direct consumption or as an input in the production of any of two services, say, y_1 and y_2. Let $u^i(y^i,x^i)$ represent the utility function of the i-th member, $i \in N$, where x^i represetnts consumption of commodity x, while $y^i = (y_1^i, y_2^i)$ represents consumption of the two services, for each $i \in N$. For simplicity we assume that $y_j^i \in \{0,\bar{y}_j\}$, $\forall\ i \in N$, where $\bar{y}_j$, $j = 1,2$, represents a positive constant.

For each coalition $c \in C$, the following table gives: (a) the quantity of x required as an input if that coalition is to provide $\bar{y}_j$ units of the j-th service to each of its members, $j = 1,2$, and (b) the benefit from cooperation (in units of x) over the situation where each member acts alone.

$\bar{y}_1$ for each $i \in c$			$\bar{y}_2$ for each $i \in c$		
c	Input	Benefit	c	Input	Benefit
{1}	60	0	{1},{2},{3}	100	0
{2},{3}	85	0	-	-	-
{1,2},{1,3}	135	10	{1,2},{1,3}	220	-20
{2,3}	185	-15	{2,3}	175	25
{1,2,3}	235	-5	{1,2,3}	330	-30

This is a case of a multi-issue cooperative game where the production of each of the two services and how much of commodity x is retained for consumption can be seen as representing each of the issues, i.e., $J = \{1,2,3\}$. Furthermore, as we can see from the above table, if we use a game $\tilde{\Gamma}$ to deal with this case the community will not be able to take full advantage of the existing opportunities. In particular, let us suppose that utility functions are identical to each other, increasing in x, and such that every member $i \in N$ prefers the consumption of $\bar{y}_j$,

$j = 1,2$, over the consumption of the amount of x required to produce $\bar{y}_j$ by that member. Then, the set of final allocations

$$(27)\quad \{y = (y_1^i, y_2^i, x^i)_{i \in N}\colon\ y_1^i = \bar{y}_1,\ y_2^i = \bar{y}_2,\ \forall\ i \in N,\ x^1 = 840,$$
$$815 \leqq x^i \leqq 825,\ \text{for } i = 2,3,\ \text{and } x^2 + x^3 = 1640\}$$

will correspond to the alternatives in $K^{\alpha}(\tilde{\Gamma})$, and $K^*(\tilde{\Gamma})$, while the corresponding to such alternatives coalition (club-) stucture is $\{\{1\},\{2,3\}\}$.

However, if we use a game Γ to deal with the above case, we can easily find a $\bar{\theta} \in \Theta$ such that the set of final allocations

$$(28)\quad \{y = (y_1^i, y_2^i, x^i)_{i \in N}\colon\ y_1^i = \bar{y}_1,\ y_2^i = \bar{y}_2,\ \forall\ i \in N,\ x^1 = 850,$$
$$815 \leqq x^i \leqq 840,\ \text{for } i = 2,3,\ \text{and } x^2 + x^3 = 1655\}$$

will correspond to the alternatives in $K^{\alpha}_{\bar{\theta}}(\Gamma)$. Furthermore, in this case, the corresponding coalition (club-) structures to such alternatives will be: (a) any $\tau \in T$, for the activity of retaining x for consumption; (b) $\tau_1 = \{\{1,2\},\{3\}\}$, or $\tau_1 = \{\{1,3\},\{2\}\}$, for the production of the vector of services $(\bar{y}_1,\bar{y}_1,\bar{y}_1)$; and (c) $\tau_2 = \{\{1\},\{2,3\}\}$ for the production of the vector of services $(\bar{y}_2,\bar{y}_2,\bar{y}_2)$.

In comparing (27) with (28) it is easily seen that the allocations in (28) are Pareto-superior to those in (27). This may come as a surprise because here $K^{\alpha}(\tilde{\Gamma}) = K^*(\tilde{\Gamma}) \neq \emptyset$ and, in general, the "core of an economy" selects Pareto-optimal allocations. However, this is just an illustration of the additional possibilities provided by a game Γ and the limitations of $\tilde{\Gamma}$ (i.e., $\tilde{Z} \subseteq Z$).

Because, in this example, a fixed quantity of each service is provided to each member in the community, and there are no externalities, it may seem that (28) has been obtained by appropriately combining the usual cores of three, appropriately defined, single-issue games one for each $j \in J$. Thus, it would appear that the first game deals with the provision of the vector $(\bar{y}_1,\bar{y}_1,\bar{y}_1)$ of the first service and the distribution of a maximum net benefit of 10 units of x, while the second game deals with the provision of the vector $(\bar{y}_2,\bar{y}_2,\bar{y}_2)$ of the second service and the distribution of a maximum net benefit of 25 units of x. Given the usual core of each of those two games, then, the third would deal with the trivial problem of distributing net final holdings of x for consumption.

This interpretation may be misleading, and, in general, this approach will not be valid for obtaining a core-concept of Γ. For example, even if we retain the assumption that the quantity of each service remains fixed, each activity cannot be dealt with independently of the others if, say, we introduce an externality.

Suppose, for example, that there are two methods for coalition {2,3} to provide each of its members with $\bar{y}_2$ units of the second service. The first method, denoted by B_1, requires the same input as above, i.e., 175 units of x. The second method, denoted by B_2, requires $175 + \delta$ units of x, for some $\delta > 0$. Let $B \in \{B_1,B_2\}$. The method selected by coalition {2,3} in producing $(\bar{y}_2,\bar{y}_2)$ affects the utility of player 1, i.e., $u^1(y^1,x^1;B)$. In particular $u^1(y^1,x^1;B_1) < u^1(y^1,x^1;B_2)$, $\forall\ (y^1,x^1)$. Everything else is the same as in the previous example.

An interpretation of this case is that, by following B_1, coalition {2,3} creates an externality that affects negatively player 1, and that externality can be eliminated if {2,3} were to follow B_2.

In this new example player 1 may follow strategies relative to either the production of the first service or the consumption of commodity x that will eliminate the externality. Of course much will depend on how large δ is, and the form of $u^1(\cdot)$. However, even without compensation to the members of coalition {2,3}, core-concepts that utilize θ^* may be such that player 1 will be able to make counterproposals against any proposal that implies the production of the second service by coalition {2,3} via method B_1.

4.3 Logrolling and Voting in Committees

M. Shubik and L. Van der Heyden (1978), in addressing the question of logrolling in voting/~~in~~ committees concluded that, in general, such a phenomenon will not occur. However, our conclusions in the proceding subsection ~~may~~ indicate that the results obtained by Shubik and Van der Heyden may be due to the particular model that they have used. As a voting game, Γ may present quite a variety of interesting possibilities not available in other cases. Below we present a variation of a "budget allocation game."

A set of voters N must decide on the level of expenditure x_j to be appropriated on each project j in a finite set J, subject to the constraint that $x_j \geqq 0$, for each $j \in J$, and $\sum_{j \in J} x_j \leqq B$, for a fixed positive B.

A project of "size" $f_j(x_j)$ will result if an amount equal to x_j is appropriated to the j-th project, while the costs relative to that project will be distributed among the voters in accordance with a cost allocation scheme $g_j(x_j)$, $g_j(x_j) = (x_j^1, \ldots, x_j^n)$.

Let l_j^i denote the level of expenditure that the i-th voter would like to see appropriated to the j-th project. Individual feasibility in this case is equivalent to setting $l_j^i \geqq 0$, and $\sum_{j \in J} l_j^i \leqq B$, for each $i \in N$. Furthermore, agreement on a common level of expenditure relative to a project $j \in J$ will be necessary for a coalition to be admissible relative to that project. Thus, for each $s \in \hat{S}^*$, for each $j \in J$, and for each $c_j \in \tau_j^s$, $(l_j^i;c_j^i) = (l_j(c_j);c_j)$, $\forall\ i \in c_j$, where $l_j(c_j)$ represents the commonly agreed upon level of expenditure on the j-th project by the members of c_j.

For each joint strategy $s \in \hat{S}^*$, the level of expenditure x_j to be appropriated to the j-th project, $j \in J$, is determined by voting rules, that specify "winning coalitions." In particular, given such rules, a coalition $c_j \in C(s)$, $s \in \hat{S}^*$, is a winning coalition relative to the j-th project, $j \in J$, if the level of expenditure that will be appropriated to that project is equal to the commonly agreed upon level of expenditure by the members of c_j, $l_j(c_j)$. Let W_j denote the set of winning coalitions relative to the j-th project, $j \in J$. Then, for each $s \in \hat{S}^*$, and for each $j \in J$ we can write

$$x_j = \begin{cases} l_j(c_j) & \text{if} \quad c_j \in W_j \cap \tau_j^s \\ 0 & \text{otherwise.} \end{cases}$$

By letting $x = (x_j)_{j \in J}$ denote the vector of appropriations corresponding to each $s \in \hat{S}^*$, as above, and by writing $f(x) = (f_j(x_j))_{j \in J}$, and $g(x) = (g_j(x_j))_{j \in J}$, the outcome associated with each joint strategy $s \in \hat{S}^*$ takes the form $\hat{\psi}(s) = (f(x),g(x))$.

With preferences defined over the set of all possible final outcomes in this form, the analysis in Section 3 can be used to address questions about logrolling by examining the vectors of coalition structures included in $C\Sigma_\mu^\lambda(\Gamma)$, defined in (21), for some $\lambda \in \Lambda$, and some $\mu \in M$. In particular, for non-trivial cases the possibility of logrolling will exist if $C\Sigma_\mu^\lambda(\Gamma) \neq C\Sigma_\mu^\lambda(\tilde{\Gamma})$.

5. Concluding Remarks

In addition to its direct use in the definition of a solution concept, each effectiveness rule ξ_λ, $\lambda \varepsilon \Lambda$, in (14) can be used for the derivation of a game in "effectiveness form" from Γ, or $\tilde{\Gamma}$. In particular, for each $\lambda \varepsilon \Lambda$, we can represent this form by a 4-tuple (N,H,ξ_λ,R), where in the case of Γ we should set $H = Z$, while in the case of $\tilde{\Gamma}$ we should set $H = \tilde{Z}$.

Derivation of cooperative games in effectiveness forms, as above, will put our analysis in parallel with that of Rosenthal (1972), and other more traditional approaches (see, e.g., Aumann and Peleg (1960), Aumann (1961, 1967)). For example, it is easily seen that as we have pointed out in the Introduction the <u>characteristic function form</u> of a cooperative game (α-derivation) will correspond to the special case of an effectiveness form where $\lambda = (\theta^o,\varepsilon^\alpha)$. However, there is a problem here because an effectiveness form derived from Γ is not necessarily the same as the corresponding effectiveness form that can be derived from $\tilde{\Gamma}$. Therefore, for each $\lambda \varepsilon \Lambda$, we can refer to (N,Z,ξ_λ,R) as the λ-<u>effectiveness form of</u> Γ, and to $(N,\tilde{Z},\xi_\lambda,R)$ as the λ-<u>effectiveness form of</u> $\tilde{\Gamma}$.

Solution concepts for cooperative games in effectiveness form are easy to define by modifying the method used in Section 3.

For example, it is evident from (18) and (19) that, for given $\lambda \varepsilon \Lambda$, and $\mu \varepsilon M$, the only justified threats that matter in each situation are those for which coalitions are λ-effective. But, then, the definition of a threat correspondence can be modified so that a threat will be justified only if a coalition is λ-effective for it, for given $\lambda \varepsilon \Lambda$. Therefore, given (N,H,ξ_λ,R), a solution concept will consist of all alternatives in H for which, the so modified threat correspondence is empty.

For the case where the games $\tilde{\Gamma}$ and Γ are used to describe cooperative decision situations in economic environments that are conceptually equivalent to a game G, derivation of different effectiveness forms, as above, can be seen as equivalent to different methods for obtaining models of <u>coalition production economies</u>. However, contrary to more standard approaches (see, e.g., Hildenbrand (1973), and Sondermann (1974)), where the "production set" of each coalition is given at the outset, in an effectiveness form it must be derived on the basis of some effectiveness rule ξ_λ, $\lambda \in \Lambda$. Furthermore, for a given economic environment, the model of a coalition production economy obtainable in this way may not be invariant with respect to Γ and $\tilde{\Gamma}$ or with respect to $\lambda \in \Lambda$.

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REFERENCES

Arrow, K.J.: "The Organization of Economic Activity: Issues Pertinent to the Choice of Market Versus Nonmarket Allocation," in Public Expenditure and Policy Analysis, ed. by R.H. Haveman and J. Margolis. Chicago: Markham, 1970.

Aumann, R.J.: "The Core of a Cooperative Game Without Side Payments," Transactions of the American Mathematical Society, 98 (1961), 539-552.

__________: "A Survey of Cooperative Games Without Side Payments," in Essays in Mathematical Economics in Honor of Oskar Morgenstern, ed. by Martin Shubik. Princeton, New Jersey: Princeton University Press, 1967.

Aumann, R.J., and M. Maschler: "The Bargaining Set for Cooperative Games," in Advances in Game Theory, ed. by M. Dresher, L.S. Shapley, and A.W. Tucker, Annals of Mathematics Studies, No. 52. Princeton, New Jersey: Princeton University Press, 1964.

Aumann, R.J., and B. Peleg: "Von Neumann-Morgenstern Solutions to Cooperative Games Without Side Payments," Bulletin of the American Mathematical Society, 66 (1960), 173-179.

Davis, M., and M. Maschler: "Existence of Stable Payoff Configurations for Cooperative Games," in Essays in Mathematical Economics in Honor of Oskar Morgenstern, ed. by Martin Shubik. Princeton, New Jersey: Princeton University Press, 1967.

Foley, D.K.: "Lindahl's Solution and the Core of an Economy With Public Goods," Econometrica, 38 (1970), 66-72.

Hildenbrand, W.: Core and Equilibria of a Large Economy. Princeton, New Jersey: Princeton University Press, 1974.

Hurwicz, L.: "Optimality and Informational Efficiency in Resource Allocation Processes," in Mathematics Methods in the Social Sciences, 1959, ed. by K.J. Arrow, S. Karlin, and P. Suppes. Stanford: Stanford University Press, 1960.

Koopmans, T.C.: Three Essays on the State of Economic Sciences. New York, New York: McGraw-Hill, 1957.

Ledyard, J.O.: "Incentive Compatible Behavior in Core-Selecting Organizations," Econometrica, 45 (1977), 1607-1621.

Muench, T.J.: "The Core and the Lindahl Equilibrium of an Economy With a Public Good: An Example," Journal of Economic Theory, 4 (1972), 241-255.

Peleg, B.: "Existence Theorem for the Bargaining Set $M_1^{(i)}$," in Essays in Mathematical Economics in Honor of Oskar Morgenstern, ed. by Martin Shubik. Princeton, New Jersey: Princeton University Press, 1967.

Rosenthal, R.W.: "Cooperative Games in Effectivenss Form," Journal of Economic Theory, 5 (1972), 88-101.

Shapley, L.S., and M. Shubik: "On the Core of an Economic System With Externalities," American Economic Review 59 (1969), 678-684.

Shubik, M., and L. Van der Heyden: "Logrolling and Budget Allocation Games," International Journal of Game Theory, 7 (1978), 151-162.

Starrett, D.A.: "A Note on Externalities and the Core," Econometrica 41 (1973), 179-183.

Sondermann, D.: "Economies of Scale and Equilibria in Coalition Production Economies," Journal of Economic Theory, 8 (1974), 259-291.

von Neumann, J., and Oskar Morgenstern: Theory of Games and Economic Behavior, Third Edition. Princeton, New Jersey: Princeton University Press, 1953 (First Edition, 1944).