99

DUALITY AND WELFARE MEASUREMENT IN THE HOUSEHOLD PRODUCTION FRAMEWORK

J.F. Scoggins
Department of Economics
University of Florida

December, 1984

In "Welfare Measurement in the Household Production Framework," Bockstael and McConnell (1983) state that "duality results that normally allow us to move between Marshallian and Hicksian functions are not applicable for commodities in the household production framework." They go on to state that both "the utility and expenditure functions exist [in the household production case], but the absence of prices prevents the use of Roy's Identity to derive the Marshallian curve from the indirect utility function." Bockstael and McConnell base their conclusions on previous work by Pollak and Wachter (1975), which proved that the commodity shadow price, the marginal cost of the last unit produced, is independent of the quantity consumed, if and only if the household's technology exhibits both constant returns to scale and no joint production.

It is the intent of this paper to show that in certain cases of household commodity production, even with non-constant returns to scale, there is a duality theory that allows us to measure exact consumer's surplus, if the production technology is known. Lemmas analogous to Shephard's lemma and Roy's identity are proven.

Introduction and Review.

The household production literature as embodied by Becker (1965), Pollak and Wachter (1975) and (1977), Barnett (1977) and Pollack (1978), was developed in the context of its relevance to the allocation of time as a joint input into the production of several commodities. The joint household production phenomenon causes a uniqueness problem in determining the Marshallian demand function.

There are cases of home production, however, where time is not a relevant factor. As Mathur (1984) points out, notable among these is the production of residential thermal comfort, which uses fuel and capital (air-conditioners, insulation, etc.) as inputs. In this case, the Pollak and Wachter Marshallian demand function for the produced commodity is unique, even though the household faces a non-linear budget constraint.

To review briefly, Pollak and Wachter derive the Marshallian demand function by specifying a utility function

$$U(z), \tag{1}$$

which is maximized subject to the budget constraint $C(r,p,z)-y = 0$. z is a vector of consumption goods that are either purchased at a fixed price or are produced from purchased inputs. r is a vector of input prices, p is a vector of consumption good prices, C is the cost function and y is income. For simplicity, I will consider only the case where

$$C(p,r,z) \;=\; C^1(r_1,r_2,z_1) + p_2z_2. \tag{2}$$

This is the case, where z_1 is produced from two inputs. The cost of producing z_1, C^1, is non-linear and z_2 is purchased at a constant price, p_2.

Constrained utility maximization results in the optimal consumption of z_1 and z_2 as functions of r_1 and r_2 (input prices), p_2 and y:

$$z_1 = f(r_1,r_2,p_2,y) \tag{3}$$

and

$$z_2 = g(r_1,r_2,p_2,y). \tag{4}$$

The Marshallian demand function for z_1 can be "traced-out" by varying the marginal cost of the last unit of z_1 produced by varying an input price, say r_1. This is equivalent to solving equation (3) for r_1 and substituting the resultant expression into the marginal cost function. The result is a "shadow price" of z_1, Π_1, as a function of z_1, r_2, p_2 and y, the Marshallian demand function,

$$\Pi_1 = D(r_2,p_2,z_1,y). \tag{5}$$

Figure (1) illustrates this process. By lowering r_1 from r_1' to r_1'', the slope of the non-linear production possibilities frontier (PPF) is decreased, thus lowering the marginal cost of producing z_1. This induces a greater quantity of z_1 to be consumed. The area labeled L, bordered by the vertical axis, the marginal cost functions and function D (the Marshallian demand function) is the change in consumer's surplus.

<u>The Expenditure Function and Hicksian Demand.</u>

To establish duality results in this case, an indirect utility function is determined,

$$U = v(f(r_1,r_2,p_2,y),g(r_1,r_2,p_2,y)). \tag{6}$$

Solving equation (6) for y results in the expenditure function,

$$y = e(r_1,r_2,p_2,U). \tag{7}$$

Bockstael and McConnell assert that this expenditure function "fails to yield comparative static results," since "the joint cost function C(z,r) will not in general be linearly homogeneous in z". Despite this fact, the expenditure function can be used to derive a Hicksian demand function and comparative statics.

Shephard's lemma, for the non-constant returns to scale case:

$$e_{r_1}(r_1,r_2,p_2,U) = C^1_{r_1}(r_1,r_2,p_2,t(r_1,r_2,p_2,U))$$

or (8)

$$t(r_1,r_2,p_2,U) = C^{1^{-1}}_{r_1}(r_1,r_2,p_2,e_{r1})$$

where e_{r_1} is the derivative of e with respect to r_1 and t is the income compensated version of equation (3).

Proof:

Let us define a function,

$$A(r_1,r_2,p_2) = e(r_1,r_2,p_2,U) - C^1(r_1,r_2,\overset{*}{z}_1) - \overset{*}{z}_2 p_2 \,, \qquad (9)$$

where $\overset{*}{z}_1$ and $\overset{*}{z}_2$ are the expenditure minimizing amounts of z_1 and z_2 that produce U, when $r_1 = \overset{*}{r}_1$, $r_2 = \overset{*}{r}_2$ and $p_2 = \overset{*}{p}_2$. Since $e(r_1,r_2,p_2,U)$ is the cheapest way to get U amount of utility, $A(r_1,r_2,p_2)$ is always nonpositive. At $r_1 = \overset{*}{r}_1$, $r_2 = \overset{*}{r}_2$ and $p_2 = \overset{*}{p}_2$, $A(\overset{*}{r}_1,\overset{*}{r}_2,\overset{*}{p}_2) = 0$. Since this is a maximum value of $A(r_1,r_2,p_2)$, its derivative must be zero,

$$A_{r_1}(\overset{*}{r}_1,\overset{*}{r}_2,\overset{*}{p}_2) = e_{r_1}(\overset{*}{r}_1,\overset{*}{r}_2,\overset{*}{p}_2,U) - C^1_{r_1}(\overset{*}{r}_1,\overset{*}{r}_2,\overset{*}{z}_1) = 0. \qquad (10)$$

Since $\overset{*}{z}_1$ is the expenditure minimizing amount of z_1, it is the income compensated quantity of z_1 produced and consumed where U is held constant,

$$\overset{*}{z}_1 = t(r_1,r_2,p_2,U). \qquad (11)$$

Thus,

$$e_{r_1}(r_1,r_2,p_2,U) = C^1_{r_1}(r_1,r_2,t(r_1,r_2,p_2,U)) \text{ or}$$

$$t(r_1,r_2,p_2,U) = C^{1^{-1}}_{r_1}(r_1,r_2,e_{r_1}). \qquad \text{Q.E.D.} \qquad (12)$$

Since t is the income compensated version of equation (3), the same method of "tracing-out" the Hicksian demand function by varying the marginal cost of z_1 can be done,

$$\Pi_1 = h(r_2,p_2,z_1,U). \tag{13}$$

This process is illustrated in figure (2). Keeping income constant, lowering r_1 increases utility from U' to U". Exact welfare measures are derived by determining the quantity of z_1 consumed when r_1 is r_1'' and utility is held at U' (Z_{10}'') and when r_1 is r_1' and utility is at U" (Z_{11}'). Area I is the compensating variation and areas I, J and K sum to be the equivalent variation. Scoggins (1984) proved that

$$\text{area}(I) = e(r_1'',r_2,p_2,U') - e(r_1',r_2,p_2,U') \tag{14}$$

and

$$\text{area}(I+J+K) = e(r_1'',r_2,p_2,U'') - e(r_1',r_2,p_2,U''). \tag{15}$$

Thus these derived exact welfare measures do conform to the standard definitions of compensating variation and equivalent variation.

Roy's Identity and Marshallian Demand.

A useful tool in welfare and demand analysis is Roy's identity. Now that Shephard's lemma has been modified, a new Roy's identity can be derived.

Roy's identity, for the non-constant returns to scale case:

$$\frac{-v_{r_1}}{v_y} = C^1_{r_1}(r_1,r_2,f(r_1,r_2,p_2,y))$$

or (16)

$$f(r_1,r_2,p_2,y) = C^{1^{-1}}_{r_1}\left(r_1,r_2,\frac{-v_{r_1}}{v_y}\right).$$

where v_{r_1} and v_y are the derivatives of the indirect utility function with respect to r_1 and y respectively.

Proof:

Suppose that z_1^* and z_2^* maximize utility at (r_1^*,r_2^*,p_2^*,y^*). This means that $z_1^* = f(r_1^*,r_2^*,p_2^*,y^*)$ and $z_2^* = g(r_1^*,r_2^*,p_2^*,y^*)$. Let $U^* = U(z_1^*,z_2^*) = U(f(r_1^*,r_2^*,p_2^*,y^*),g(r_1^*,r_2^*,p_2^*,y^*)) = v(r_1^*,r_2^*,p_2^*,y^*)$. Then it is also true that

$$U^* = v(r_1,r_2,p_2,e(r_1,r_2,p_2,U^*)). \tag{17}$$

Since U^* is the maximal utility, we can differentiate it to get:

$$0 = v_{r_1}(r_1^*,r_2^*,p_2^*,y^*) + v_y(r_1^*,r_2^*,p_2^*,y^*)e_{r_1}(r_1,r_2,p_2,U^*)$$

or

$$e_{r_1}(r_1,r_2,p_2,U^*) = \frac{-v_{r_1}}{v_y}. \tag{18}$$

From the modified Shephard's lemma,

$$e_{r_1}(r_1,r_2,p_2,U^*) = C^1_{r_1}(r_1,r_2,t(r_1,r_2,p_2,U^*)). \tag{19}$$

Since at maximum utility,

$$t(r_1,r_2,p_2,U^*) = f(r_1,r_2,p_2,y), \tag{20}$$

we see that

$$C^1_{r_1}(r_1,r_2,f(r_1,r_2,p_2,y)) = \frac{-v_{r_1}}{v_y}$$

or (21)

$$f(r_1,r_2,p_2,y) = {C^1_{r_1}}^{-1}(r_1,r_2,\frac{-v_{r_1}}{v_y}). \qquad \text{Q.E.D.}$$

Conclusion.

Welfare analysis of the household production of commodities has several complications not found in market supplied commodities. One of these is a non-linear budget constraint. If the production technology and its related cost function are known, both Marshallian and Hicksian demand functions can be derived, along with related welfare measures.

This analysis has particular importance in the study of energy demand and conservation programs. Alterations of input prices (i.e., price of insulation) have welfare impacts, which can be measured using the above duality theory.

References.

Barnett, W.A., "Pollak and Wachter on the Household Production Function Approach," Journal of Political Economy, October 1977, 85, 1073-82.

Becker, G.S., "A Theory of the Allocation of Time," Economic Journal, September 1965, 75, 493-517.

Bockstael, N.E., and McConnell, K.E., "Welfare Measurement in the Household Production Framework," American Economic Review, September 1983, 73, 806-14.

Mathur, V.K., "Effectiveness of Building Energy Performance Standards to Curtail Household Energy Demand: A Theoretical Analysis," The Energy Journal, January 1984, 5, 99-114.

Pollack, R.A., "Welfare Evaluation and the Cost-of-Living Index in the Household Production Model," American Economic Review, June 1978, 68, 285-99.

________ and Wachter, M.L., "The Relevance of the Household Production Function and Its Implications for the Allocation of Time," Journal of Political Economy, April 1975, 83, 255-77.

__________ and __________, "Reply: Pollack and Wachter on the Household Production Function Approach," Journal of Political Economy, October 1977, 85, 1083-86.

Scoggins, J.F., "Utility Sponsored Conservation Investment Programs," mimeo, December 1984, University of Florida.

Varian, H., Microeconomic Analysis, New York, 1978.