Efficient Market-Clearing Prices in Markets with Nonconvexities

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Abstract

This paper addresses the existence of market clearing prices and the economic interpretation of strong duality for integer programs in the economic analysis of markets with nonconvexities (indivisibilities). Electric power markets in which nonconvexities arise from the operating characteristics of generators motivate our analysis; however, the results presented here are general and can be applied to other markets in which nonconvexities are important. We show that the optimal solution to a linear program that solves the mixed integer program has dual variables that: (1) have the traditional economic interpretation as prices; (2) explicitly price integral activities; and (3) clear the market in the presence of nonconvexities. We then show how this methodology can be used to interpret the solutions to nonconvex problems such as the problem discussed by Scarf (1994).

Economics, Equilibrium Pricing; MIP models of markets, MIP Applications

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I. Introduction

Scarf (1990, 1994) describes most markets in today's advanced economies as having considerable indivisibilities (nonconvexities). For example, firms must make discrete decisions on whether to invest in a new project or when to start-up a production process. It has been widely believed that in the presence of nonconvexities in the cost function, it is not possible to guarantee the existence of linear prices that will allow the market to clear, unless the solution to the relaxed convex problem just happens to produce an integral solution (e.g. assignment problems).

Unfortunately, the modeling of nonconvexities such as discrete choices and economies of scale have largely been avoided due to the intractability of such problems. Standard graduate texts in microeconomics such as Kreps (1990) and Varian (1992) note that assuming away nonconvexities is unrealistic, but they proceed with the standard assumptions without addressing the issue further. Mathematical references used by economists such as Chiang (1984) and Takayama (1985) do not mention integer programming for solving optimization problems with nonconvexities. In the face of nonconvexities, linear *commodity* prices in general will result in either a situation of excess supply or excess demand, and the market will not clear. As a simple example in which nonconvexities prevent a market from clearing, consider a market in which all firms have the same cost and entry is free. Each firm must incur a fixed cost of one to produce any positive amount of a good in the range (0,1]; in that range, marginal cost is zero. If the market demand curve is P = 2 – 0.6Q, then there is no market equilibrium. For any price less than 1, no firm will produce and there will be a shortage. For any price strictly greater than 1, quantity supplied is infinite, and there is a surplus. Finally, for P =1, quantity demanded is 1.67, but the quantity supplied will be no more than 1, because if a second firm enters, it will not earn enough revenue to cover its fixed cost.

Because of such problems, it has been more convenient and tractable to employ linear or convex nonlinear optimization models to represent profit maximization problems for producers and utility maximization problems for consumers. Such optimization problems assume desirable properties such as the continuity and concavity of the objective function to be maximized, and the convexity of the feasible region defined by the constraint set. As a justification for the assumption of convexity, Arrow and Hahn (1971), Mas-Colell *et al.* (1995), Takayama (1985), and Varian (1992) argue that if agents in an economy were replicated many times, then linear prices will support a competitive equilibrium. Arrow and Hahn use the convex hull of the non-convex set of constraints to show an "approximate" equilibrium. An equilibrium in such a market yields a linear commodity price (or vector of linear prices) and quantity (or vector of quantities) such that all economic agents maximize their objectives and the market clears (the quantity supplied equals the quantity demanded for each commodity priced). Conceptually, a linear price vector arises out of the application of the Separating Hyperplane Theorem. (For example, see Takayama 1985, pp. 39-49, 103). Such simplifying assumptions about the objective function and constraint sets allow economists to prove the existence of market clearing prices using fixed-point arguments. Computationally, if the market equilibrium problem is solved by Samuelson's (1952) principle, the equilibrium prices for such markets are simply the dual variables (shadow prices or LaGrange multipliers) for the market clearing constraints of the goods.

Such modeling assumptions have allowed economists to construct useful models of economic behavior and to conduct insightful simulation experiments with these increasingly complex models. But since the work of Gomory and Baumol (1960), analogous dual variable interpretations for mixed-integer programs have eluded economists and mathematicians. As an example, Geoffrion and Nauss (1977) state "(integer programming) models have no shadow prices or dual variables with an interpretation comparable to that in linear programming." The economic literature continues to reflect this belief. Current market models are largely unable to deal with the significant nonconvexities that actually exist. For example, whether or not to invest in a new capital project or whether or not to start-up a production operation are discrete decisions. Many production processes have economies of scale, a property contrary to the linearity/convexity assumption. The nonexistence of market clearing prices can be a real problem, and some degree of centralized coordination may be required in some markets to reach the welfare maximizing solution.

An important market where nonconvexities are significant and are a concern in constructing prices is the short-term (day- to week-ahead) electric power market. Nonconvexities include start-up and shut-down costs along with minimum output requirements (which state that if a plant is running, it must produce at least a certain amount). The lumpiness of the costs in these markets can have a large influence on operating schedules and ultimately investment. It is widely noted that the presence of nonconvexities implies that there will be no linear commodity prices that will support an equilibrium (e.g., Johnson *et al.*, 1997, Madrigal and Quintana, 2000; Hobbs, Rothkopf *et al.*, 2001). This lack of prices leads to a potential mismatch of supply and demand that is of concern to the engineers responsible for maintaining system balance and stability, to the economists and market designers who are interested in promoting market efficiency, and to the market participants themselves who are worried about how steps taken to balance supply and demand might affect their outputs and revenues.

In this paper, we present a method for constructing a set of linear prices that will support a Walrasian competitive equilibrium in markets with nonconvexities that is based on mixed integer programming (MIP). Prices are derived from solving a MIP and an associated linear program and have a corresponding analogy to non-linear (multi-part) prices. These prices will support equilibrium allocations in a decentralized auction-based market. That is, if a Walrasian auctioneer announced to market participants the prices we derive, from within the set of the maximizing allocations chosen by agents is a set that would clear the market. (For any ties that might occur, the auctioneer would randomly select winners.)

The role of non-linear pricing in markets with non-convexities has been recognized and well researched (See Wilson 1993). However, the market environments in which non-linear prices have been examined have not been considered perfectly competitive. For example, monopoly utility services often are subject to non-linear pricing in the form of a demand or access charge plus a variable linear charge for the service. Additionally, non-linear prices can be employed to enable firms to capture rents through price discrimination or to strategically compete for certain segments of the market. In contrast, the prices we propose, while being analogous to non-linear prices, can be employed in a competitive environment where market agents are price takers and therefore do not have any market power and do not compete strategically.

Our method for calculating equilibrium prices is straightforward. First, we solve a MIP to find the optimal allocation. Next, we remove the integrality constraints and insert equality constraints (cuts) that force the integer variables to assume their optimal values in the resulting linear program (LP). We then solve the LP to find the associated dual prices on the market clearing conditions and added equality constraints. These dual (or shadow) prices then can be used as prices to support a competitive equilibrium.

The paper proceeds as follows. Section II reviews the relevant literature. Then in Section III, we define a linear program that solves mixed-integer programs and discuss why linear prices on the output commodity are not sufficient for a competitive equilibrium in the face of non-convexities. In Section IV, we discuss the example used by Scarf (1994), and we show how the market clearing prices can be computed for his model. In Section V, we provide a general formulation of the market clearing model and a general proof that demonstrates that we can always find prices that will clear a market with indivisibilities, so long as we can find the optimal solution to the MIP that describes the market. Section VI concludes and discusses some applications and extensions.

II. Related Literature

The economics and management science literature has occasionally addressed the problem of finding dual price interpretations to integer programs and MIPs. The classic work in this area is Gomory and Baumol (1960). In order to find the solution to the MIP, Gomory and Baumol add additional constraints or cutting planes (to the LP relaxation of the MIP), which in their case they define as linear combinations of existing constraints, until the solution to the augmented LP results in an integer solution. With this methodology, they obtain shadow prices that are non-negative, impute zero profits, and infer zero prices for activities not used to capacity.

However, the shadow prices obtained by Gomory and Baumol have some peculiar properties. The prices themselves are integer valued and can vary with the choice of additional constraints. Gomory and Baumol refer to the additional constraints needed to solve the problem as “artificial”, and they refer to the shadow prices on the additional constraints as “artificial capacity prices” or as the “opportunity costs of the indivisibilities.” Moreover, they observe that constraints in the non-integer solution that have positive prices may have zero prices in the integer solution. For example, a warehouse may have a capacity of, say, 3.4 units, but the units only come in integer values. In this case, the capacity constraint may be binding (by making 4 units infeasible), but there is still positive slack. In an economic sense, there should be a positive price associated with this constraint.

In an attempt to deal with these peculiarities, Gomory and Baumol attempt to impute the prices from the “artificial” constraints back into the original constraints to get prices. These recomputed prices have the property that they will yield zero profit and any good with a zero price is truly a free good in an economic sense. Unfortunately, the recomputed prices may not price at zero all free goods.

The Gomory and Baumol prices also have some welfare implications. First, competitive output combinations arising from these prices will be efficient. However, Gomory and Baumol go on to state:

> "Unlike the ordinary linear programming case, however, not every efficient output can be achieved by simple centralized pricing decisions or by competitive market pricing processes. Moreover, it is possible in the integer programming case that there exists no hyperplane which separates the feasible lattice points from those which are preferred to or indifferent with the optimal lattice point. In other words, there may exist no set of prices which simultaneously makes the optimal point, Q, the most profitable among those that can be produced and the cheapest among those that consumers consider to be at least as good as Q. That is, at any set of prices either producers will try to make, or consumers will demand, some other output combination" (p. 537).

It is important to note here that Gomory and Baumol are searching for linear, uniform commodity prices. They do note that there are decentralized discriminatory prices that would lead to an efficient allocation, but they do not pursue this line of inquiry further.

Additional relevant research stems from Shapley and Shubik (1972) and their discussion of assignment markets. They point out that linear commodity prices that support an equilibrium are available in a market with indivisibilities when the market can be modeled as a two-sided assignment game. When this is done, the dual variables of the resulting assignment problem can be used to create prices that clear the market. While Shapley and Shubik lay the groundwork for deriving prices in markets with indivisibilities, their approach is successful only when the linear programming relaxation coincidently solves the integer programming representation of the market, as is the case with the assignment problem. While this approach addresses some markets with indivisibilities, it does not address the general case where the indivisibilities arise from such things as startup costs and economies of scale.

Later authors have addressed variations on the assignment game (Leonard, 1983; Bikhchandani and Mamer, 1997; Bikhchandani and Ostroy, 2001a; and Bikhchandani and Ostroy, 2001b). The central issue in these papers is that equilibrium supporting prices can be obtained when the underlying market is represented as an assignment problem. In these cases, there are a set of prices (dual variables) for the commodities that fall in the core of an assignment game, and there may be, and generally are, many pricing vectors that support an equilibrium. All of the researchers stop short of the next step, addressed in this paper: the problem of deriving a set of prices that will support a Walrasian competitive equilibrium in a general nonconvex market (i.e. one where the LP relaxation fails to coincidently solve the MIP). In particular, Bikhchandani and Ostroy 2001b extend results for package bidding beyond the assignment model, but not to general MIPs.

Scarf (1990, 1994) describes the simplex algorithm for solving LPs as being analogous to the economic institution of competitive markets, specifically a Walrasian auction. The similarities are that in a Walrasian auction, the auctioneer calls out prices until markets clear and there are zero profits, while the simplex algorithm attempts candidate solutions until no activity or slack variable can be introduced into the solution basis that improves the solution. Scarf then goes on to note that once increasing returns to scale or indivisibilities are introduced, it is difficult to draw any similar analogies between integer programming algorithms and firms or markets with such indivisibilities. Moreover, Scarf (1990) makes the following observations:

> "And, perhaps even more significant for economic theory, none of these algorithms seemed capable of being interpreted - by even the most sympathetic student - in meaningful economic terms. ... This test (for convex programs) for optimality is not available for integer programs; there simply need not be a set of prices that yields a zero profit for the activities in use at the optimal solution. ... Is its profitability at the equilibrium prices a necessary and sufficient condition for a Pareto improvement - for the possibility that everyone can be made better off using this new activity? The answer, unfortunately, is no! ... The market test fails because the firm, whose technology is based on an activity-analysis model with integral activity levels, cannot be decentralized without losing the advantages of increasing returns to scale" (p. 381-382).

In an attempt to link integer-programming algorithms to economic institutions Scarf (1990) draws the analogy of the internal structure of a large firm to an integer-programming algorithm. Scarf looks at an integer-programming algorithm that breaks the large integer program down into a decision tree in which smaller sub-problems can be solved in polynomial time. Scarf likens the branches of this tree to divisions of a large firm, and the nodes as managers making decisions for each of the branches below it. However, Scarf (1990) offers no method for computing prices that will help clear the market in the presence of indivisibilities, and that will provide a pricing test for Pareto improvements.

More recently, Williams (1996) discusses the mathematics of duality and its potential economic interpretations. Williams observes the same problems encountered by Gomory and Baumol in that there are often binding constraints in integer programs that have positive slack. Williams laments that this problem leads to mathematical difficulties, particularly a violation of complementarity conditions. Moreover, Williams contends that the dual prices found by Gomory and Baumol do not provide a proof of optimality (equality of primal and dual objective functions). Williams then proposes a dual as a more complex extension of Gomory and Baumol. Computation of the dual relies heavily on advanced integer optimization techniques, and in general it is difficult to associate any dual variables with a particular resource. Finally, the dual proposed by Williams, while providing a proof of optimality, still does not satisfy complementarity conditions.

Other operations researchers have also attempted to define interpretable and computable duals/shadow prices/price functions for integer programs (Wolsey, 1981). For instance, Crema (1995) defines a shadow price based on the average incremental contribution of a resource, while Williams (1989) defines a marginal value as the directional partial derivative of the optimal objective function value with respect to perturbations in the right hand side.

The approach described here models markets with indivisibilities first as a MIP and then as a linear program that is created from the optimal solution to the MIP. In terms of the assignment markets literature and traditional microeconomic theory, we are expanding the set of commodities in the market (and therefore commodities to be priced) by at most one extra good for each indivisibility. In this vein, the market can be thought of as a pseudo-assignment game for the indivisibilities combined with a continuous game for the commodities. In the vernacular of game theory, the general non-convex market is a game that has an empty core in the initial commodity space. By expanding the game to include the indivisibilities as additional commodities, the game is converted to one in a higher dimensional space where there is a nonempty core. As a result, this expanded game can always be solved to produce a set of linear prices for indivisibilities and commodities that supports a competitive equilibrium and clears the market.

III. Prices in the LP that Solve the MIP

A mixed integer problem with m continuous variables and n integer variables (R^m x Z^n) that has a feasible and bounded optimal solution can be converted to a linear program with at most $m+n$ continuous variables (R^{m+n}) and at most n additional linear constraints (Gomory and Baumol, 1960). These statements can be proved by observing that an additional constraint can be defined for each integer variable setting the variable equal to its optimal value, which produces a LP that solves the MIP. (It is worth noting here that simply solving the integer program and inserting the optimal values as equality constraints is not what Gomory and Baumol had in mind. They were primarily concerned with using cutting planes to find the solution to the integer program. As the reader will see below, we separate the issues of finding the optimal solution and identifying cuts whose duals can be interpreted as prices.) Thus, n can be thought of as the maximum number of additional degrees of freedom needed to price the output or the maximum additional dimensions needed for the space where the separating hyperplane or linear support function exists. In R^m, the support function is nonconvex and poorly behaved (Gould 1971). In R^{m+n}, there is always a separating hyperplane.

The next challenge is to find an economic interpretation of the linear prices in R^{m+n}. For convex problems there is a commodity vector for which there is a corresponding price vector. In the fixed charges example of Gomory and Baumol (1960, pp. 538-540), they assume that the additional dimensions are artificial and not meaningful. However, we believe the additional dimensions required for integer problems can be usefully viewed as additional commodities. One can think of the sub-optimality associated with integral activities and linear prices as a misspecification of the commodity space. If start-ups, or any other integral activity, are necessary for production, the auctioneer can consider these activities as separate commodities complementary to the output commodity production activities that can therefore be priced as well.

IV. Scarf's Example

As an example of a market with non-convexities that lacks a market-clearing price for the commodity, consider the problem put forth by Scarf (1994). He postulates two types of plants, each with significant fixed costs and relatively small marginal costs (Table 1). The objective of the problem (auctioneer) is to minimize the total cost of satisfying a fixed level of demand. The corresponding decentralized market problem would be for each plant of each type to maximize profits subject to internal constraints and satisfy market feasibility.

Suppose that we were to attempt to satisfy a fixed demand of 61 units. The optimal solution to this problem would be to build three Smokestack plants and two High Tech plants with each running at full capacity except for the last Smokestack plant that only produces 15 units. What prices would support a competitive equilibrium in the decentralized market problem? In the context of linear prices, candidate prices might include the marginal production costs of each type of plant and the average costs at full capacity of each type of plant. Yet if price equaled either of the marginal costs (2 or 3), neither type of technology would want to produce. Each type would incur losses, so it is profit maximizing at those prices to neither build nor produce. But on the other hand, if the price equaled the average cost of the Smokestack technology at capacity (6.3125), then two Smokestack type plants would be making zero profits, and the third Smokestack plant would be operating at a loss. At this price, the High Tech types would be making positive profits, and an infinite number of this type would want to enter the market. Therefore, 6.3125 cannot be an equilibrium price since there would be excess supply at this price. The only other serious candidate price is the average cost of the High Tech type at capacity (6.2857). At this price the High Tech types would make zero profit if they operate at full capacity, but the Smokestack types would still incur losses. Therefore, no Smokestack types would wish to enter; further, if enough High Tech types enter to meet the demand of 61, the last unit would not be operating at capacity, and would be incurring losses. Thus, a price of 6.2857 cannot be an equilibrium either.

Table 1. Production Characteristics: Smokestack versus High Tech (from Scarf, 1994)

Characteristic	**Smokestack (Type 1 Plant)**	**High Tech (Type 2 Plant)**
Capacity	16	7
Construction Cost	53	30
Marginal Cost	3	2
Average Cost at Capacity	6.3125	6.2857
Total Cost at Capacity	101	44

Now, consider the construction (start-up) for each type as a separate commodity so that there are now three commodities that must be priced: the final output, construction of the Smokestack type, and construction of the High Tech type. Let a price of 3, the marginal cost of the higher cost type, be the candidate price for the final output. Let a price of 53, the construction cost of the Smokestack type, be the candidate price for building the Smokestack type. Finally, let a price of 23 be the candidate price for building High Tech types. A price of 3 on the final output makes sense, in the example above, since the third Smokestack unit can produce one more unit at a marginal cost of 3 before being at capacity. At the candidate prices, all Smokestack units would receive a price of 3 for the final output that they can produce at a marginal cost of 3. Each Smokestack unit then receives a price of 53 for construction, leaving each Smokestack unit with zero profits. The High Tech units each receive a price of 3 for the final output that they can produce at a marginal cost of 2, leaving each High Tech unit with a margin of 1 per unit of output. At the candidate construction price of 23, each High Tech unit is left with precisely zero profit. Note that the construction price that High Tech units receive is not equal to its actual construction costs. If the market were to naively pay them actual construction costs, the High Tech units would be making positive profits that would lead to entry of an infinite number of High Tech units and excess supply.

Thus, if entry decisions in that example are viewed as commodities, competitive equilibrium supporting prices can be constructed. It turns out that these prices are the dual variables for a linear program augmented by two cuts that define the number of Smokestack and High Tech units as equaling 3 and 2, respectively. In the remainder of this section, we analyze Scarf's problem further and then present the original mixed integer programming formulation along with the augmented LP that solves it.

Table 2. Cost Minimizing Choices of Plants and Output Levels (from Scarf 1994)

Demand	Type 1 Plants (Smokestack)	Type 2 Plants (High Tech)	Output of Type 1	Output of Type 2	Total Cost
55	3	1	48	7	347
56	0	8	0	56	352
57	1	6	15	42	362
58	1	6	16	42	365
59	2	4	31	28	375
60	2	4	32	28	378
61	3	2	47	14	388
62	3	2	48	14	391
63	0	9	0	63	396
64	4	0	64	0	404
65	1	7	16	49	409
66	2	5	31	35	419
67	2	5	32	35	422
68	3	3	47	21	432
69	3	3	48	21	435
70	0	10	0	70	440

Table 2 presents the least-cost solutions for demands ranging from 55 to 70 units in the example presented by Scarf (1994). We calculated market-clearing prices for these problems using the following procedure:

1. Formulate the problem as a mixed integer program and solve.
2. Find a LP that solves the MIP by adding cuts that set the integer variables to their optimal values.
3. Use the dual variables and primal quantities from the linear program to form an efficient contract.

A MIP formulation of the Scarf problem to find the cost minimizing set of units and outputs is:

Minimize: $$\sum_i (53z_{1i} + 3q_{1i}) + \sum_j (30z_{2j} + 2q_{2j}) \qquad (4.1)$$

subject to: $$\sum_i q_{1i} + \sum_j q_{2j} = Q \qquad (4.2)$$

$$-16z_{1i} + q_{1i} \le 0 \qquad \forall i \qquad (4.3)$$

$$-7z_{2j} + q_{2j} \leq 0 \qquad \forall j \qquad (4.4)$$

$$q_1, q_2 \geq 0 \qquad \forall i, j \qquad (4.5)$$

$$z_{1i}, z_{2j} \in \{0,1\} \qquad \forall i, j, \qquad (4.6)$$

where:

z_{1i} and z_{2j} represent the decision to start up plant i (i = 1,2,…, I) of I available Smokestack plants and plant j (j=1,2,…,J) of J High Tech plants, respectively, and

q_{1i} and q_{2j} are the quantities of output for Smokestack plant i and High Tech plant j, respectively.

Note that the formulation shown here can take an unnecessarily long time to solve unless modern MIP software is used (Hobbs, Stewart, et al., 2001). An equivalent formulation that would solve more quickly on basic MIP solvers defines z_1 and z_2 as representing the total numbers of units of types 1 and 2, respectively, and q_1 and q_2 as representing their respective outputs.

A linear program that solves the above MIP can be created by replacing the integer constraint (4.6) with the two sets of constraints:

$$z_{1i} = z_{1i}^* \qquad \forall i \qquad (4.7)$$

$$z_{2j} = z_{2j}^* \qquad \forall j \qquad (4.8)$$

where z_{1i}^* and z_{2j}^* are the optimal values from the MIP. The dual variables for constraints (4.2), (4.3), (4.4), (4.7), and (4.8) in the LP are denoted by the symbols y, y_{1i}, y_{2j}, w_{1i}, and w_{2j}, respectively, and they represent, in order, the single *commodity* price for each output unit produced, the capacity price for the ith Smokestack plant, the capacity price for jth High Tech plant, the start-up (construction) price for the ith Smokestack plant, and the start-up (construction) price for the jth High Tech plant.

Table 3 summarizes the values of the dual variables from solving the LP for each of the demand instances in Table 2. As we show in the next section, the dual variables for the market constraint (4.2) and integer variable constraints (4.7) and (4.8) collectively can be used by a market operator (auctioneer) to define a set of prices that clear the market and are efficient. Each Smokestack plant is paid $w_{1i}^*z_{1i}^*$ for starting up and $y^*q_{1i}^*$ in exchange for producing q_{1i}^*, and each High Tech plant is paid $w_{2j}^*z_{2j}^*$ for starting up and $y^*q_{2j}^*$ in exchange for producing q_{2j}^*.

In general, it is necessary to specify the quantity to be produced in the contract because price signals alone as decentralized mechanisms are not always sufficient to clear the market for either convex or nonconvex problems. In convex optimization, only cost functions that are strictly convex at the equilibrium will, in general, allow for pure price signals in an auction context. Otherwise, if a supplier is on the flat part of a marginal cost curve, the auctioneer must send quantity signals in addition to price signals to obtain a feasible solution that clears the market without excess supply or demand.

Negative prices are payments to the auctioneer by a plant as part of choosing to produce). (In this case, the capacity price is implicitly embedded in the start-up price. For Type 2 units, the derivative of the Lagrangean defined under the LP with respect Z_{2j} and setting that equal to zero is $30 - 7y_{2j} - w_{2j} = 0$.) These prices yield nonnegative profits for each plant. No plant prefers to change its output and start-up decision at the prices that have been announced. Under these prices, those producing and those not producing are both economically satisfied with their chosen production allocation, in the sense that under the announced prices, no other levels of output would increase profit. Finally, the solution is efficient (in this case, least cost).

Table 3: Dual Prices for Scarf's Problem

		Plant 1 (Smokestack)		**Plant 2 (High Tech)**	
Dual Price Set	**Commodity Price (y)**	**Start-up Price(w_{1i})**	**Capacity Price(y_{1i})**	**Start-up Price(w_{2j})**	**Capacity Price(y_{2j})**
Set I[a]	3	53	0	23	-1
Set II[b]	6.3125	0	-3.3125	-.1875	-4.3125
Set III[b]	6.2857	.429	-3.2857	0	-4.2857

a. Applies to all integer demand levels of Q from 55 to 70

b. There are alternative dual solutions for demands of 55,56,58,60,62,63,64,65,67,69,70, when all units started, of either type, are operating at full capacity.

In the examples, the commodity price (y^*) is either the variable cost of unit 1 (the highest unit marginal operating cost), the average cost of unit 1 at full output, or the average cost of unit 2 at full output (Table 3). The prices may not be unique, depending on the level of demand, as indicated in Table 3. These result from degeneracy in the primal LP, stemming from the coincidence that demand exactly equals the sum of the capacities of the units in the solution. However, each set of prices yields the same revenue and output result. Each also has an economic interpretation. There are three sets of prices corresponding to the dual variables in Table 3. A example using dual price set I would be:

1. For Smokestack units: produce q_{1i}^*; get paid \$3/unit of produced (the highest marginal cost of a running unit); and get paid \$53 to start.
2. For High Tech units: produce q_{2i}^*; get paid \$3/unit of production; and get paid \$23 to start.

In this example, it turns out that all units producing are offered prices that pay exactly their costs. The start-up price is the difference between total cost and the commodity revenues. But in general, profits (scarcity rents) can be positive if, for instance, some firms possess uniquely low cost technologies and there are a limited number of plants of a specific technology. In a more general case l, there would be a finite number of plants, each with different costs. Under these conditions, many of the bidders could expect to see positive profits. In Scarf's example there are an infinite number of potential entrants with costs identical to firms in the solution. So for these entrants to be satisfied and for there to be a competitive equilibrium, no firms in the solution can be earning positive rents.

The linear prices we derive can be viewed as being analogous to multi-part prices for output commodities alone, except the prices we propose explicitly treat the non-convexity as a commodity and price it as such. Moreover, the linear prices we derive, are similar to a solution to a cooperative bargaining problem (Luce and Raiffa 1957) and to optimal multi-part pricing for natural monopolies such as demand and commodity charges in regulatory contracts (Brown and Sibley 1986). For example, a start-up price can be viewed as being similar to a demand or customer charge in natural-monopoly utility pricing. In contrast, in the presence of demand with non-zero elasticity, the best one-part prices are Ramsey (1927) prices and are "second best" when compared with efficient multi-part prices (In addition to Ramsey, see Sharkey 1982) or, as we show below, the linear prices we derive when we explicitly take integral activities as commodities. It then should come as no surprise that more degrees of freedom for pricing allows for greater efficiency in a market with no transaction costs. For MIPs, the pricing degrees of freedom needed are bounded by the sum of the number of explicit constraints and the number of integer variables. In our experience in solving electric power unit commitment models, the number of non-zero prices associated with start-up and shut-down decisions is one to two orders of magnitude smaller than the number of such variables. However, in general, the number of additional prices could, in theory, rise to the number of integer variables.

Demand as well as supply can have significant non-convexities. For example, the electricity consumption of an aluminum smelter or a cyclotron may be an all-or-nothing choice. If the buyers' problems can also be modeled in the MIP, equilibrium prices that are optimal can be devised for both buyers and sellers. In the next section, we present our general results for all markets that can be represented by mixed integer programs.

V. General Formulation and Proofs

In this section, we present a result concerning the equivalence of a MIP and an LP augmented with certain defined cutting planes. We then define a contract that an auctioneer might offer that is efficient and that has prices that support a market clearing equilibrium. Although these results are phrased as if they apply only to formal auction markets, they are also applicable to other markets.

Consider an auction market that can be represented by a **P**rimal Mixed **I**nteger **P**rogram (PIP). The formulation below assumes that the auctioneer is buying and/or selling a set of goods, and has an objective of maximizing the value to bidders. The auctioneer is simply a computer code that finds a solution to the problem:

PIP Maximize: $v_{PIP} = \sum_k c_k x_k + \sum_k d_k z_k$

Subject to: $\sum_k A_{k1} x_k + \sum_k A_{k2} z_k \le b_0$

$$B_{k1} x_k + B_{k2} z_k \le b_k \qquad \forall k$$

$$x_k \ge 0 \qquad \forall k$$

$$z_k \in \{0,1\}^{n(k)} \qquad \forall k,$$

where

x_k, z_k are commodities or column vectors of commodities for participant ($k \in K$) in the market,

c_k, d_k are the values (benefits or costs) (scalars or vectors) associated with the activities of participant k ($c_k x_k + d_k z_k$ is the benefit or cost accruing to participant k),

A_{k1}, A_{k2}, are matrices whose coefficients reflect production or demand characteristics of bidders such that the constraint $\sum_k A_{k1} x_k + \sum_k A_{k2} z_k \le b_0$ represents the market clearing constraint,

B_{k1},B_{k2} are matrices whose coefficients reflect restrictions on the individual bidders operations (e.g. production of a particular plant is limited to the capacity of that plant),

b_k represents the right hand sides of internal constraints of market participant k (scalars or column vectors),

b_0 represents output commodities to be auctioned by the auctioneer (a scalar or column vector) whose elements are different from zero in a one-sided auction and equal to zero if a two sided auction.

Lower case characters represent scalars or vectors; upper case characters represent matrices; all multiplication is of compatible dimensions.

Note that these problems may be hard to solve because, with a few special exceptions, they are NP-hard problems (*see* Johnson *et al.* (1997)).

A **P**rimal **L**inear **P**rogram that solves PIP is:

PLIP(z*) Maximize: $v_{PLIP} = \sum_k c_k x_k + \sum_k d_k z_k$

subject to: $\sum_k A_{k1} x_k + \sum_k A_{k2} z_k \le b_0$

$$B_{k1} x_k + B_{k2} z_k \le b_k \qquad \forall k$$

$$z_k = z_k^* \qquad \forall k$$

$$x_k \ge 0 \qquad \forall k \text{ ,}$$

where z_k* represents the values of the z_k variables in an optimal solution to PIP. In general, PLIP contains more constraints than PIP; these are needed for the LP to solve the MIP and to yield strong duality. The dual of PLIP(z*) is:

DLIP(z*) Minimize: $v_{DLIP} = y_0 b_0 + \sum_k y_k b_k + \sum_k w_k z_k^*$

subject to:

$$y_0 A_{k1} + y_k B_{k1} \geq c_k \qquad \forall k$$

$$y_0 A_{k2} + y_k B_{k2} + w_k \geq d_k \qquad \forall k$$

$$y_0 \geq 0$$

$$y_k \geq 0, \qquad \forall k$$

$$w_k \quad \text{unrestricted} \qquad \forall k,$$

where y_0, y_k, w_k are the dual variables, either scalars or appropriately dimensioned row vectors.

Theorem 1: $v_{PIP}* = v_{PLIP}* = v_{DLIP}*$, where * indicates the optimal solution value for the respective problems.

Proof: $v_{PIP}* = v_{PLIP}*$ because PLIP is PIP with the additional constraints that the integer variables are constrained to their optimal values (which then allows the integrality condition $z_k \in Z_k$ of PIP to be dropped as redundant). $v_{PLIP}* = v_{DLIP}*$ by strong duality of linear programs. □

With the auctioneer's problem defined and the establishment of Theorem 1, we can now define the decentralized market problem so that we can show that the dual variables (shadow prices) that constitute the optimal solution to $v_{DLIP}*$, are prices that support a competitive equilibrium. Let p_0 be the price of the output commodity, let p_k^z be the price of the commodity representing the integral activity for agent k. Then each agent k solves the following problem:

PIP_k Maximize: $v_{PIPk} = (c_k x_k + d_k z_k) - p_0(A_{k1} x_k + A_{k2} z_k) - p_k^z z_k$

subject to: $B_{k1} x_k + B_{k2} z_k \leq b_k$

$$x_k \geq 0,$$

$$z_k \in Z^k$$

With each agent's problem defined, we can now define a competitive equilibrium for the market.

Definition 1: A **competitive equilibrium** for this market is a set of prices $\{p_0^*, p_k^{z*}\}$ for all k, and allocations $\{x_k^*, z_k^*\}$ for all k such that

1. At the prices $\{p_0^*, p_k^{z*}\}$, the allocations $\{x_k^*, z_k^*\}$ solve PIP_k for all k, and
2. The market clears: $\sum_k A_{k1}x_k + \sum_k A_{k2}z_k \leq b_0$.

Theorem 2: Let $\{x_k^*, z_k^{*\prime}\}$ be the solution to PIP(z*) and PLIP(z*) and let $\{y_0^*, y_k^*, w_k^*\}$ be the solution to DLIP(z*). If $y_0^* = p_0$ and $w_k^* = p_k^z$, then the prices $\{y_0^*, w_k^*\}$ and allocations $\{x_k^*, z_k^{*\prime}\}$ for all k is a competitive equilibrium. .

Proof: We use the notation $z_k^{*\prime}$ to distinguish the optimal value of the variable z_k from the fixed right hand side of the constraint $z_k = z_k^*$ in PLIP(z*). The Karesh-Kuhn-Tucker conditions for optimality of these problems are:

$$0 \leq (y_0^*A_{k1} + y_k^*B_{k1} - c_k) \perp x_k^* \geq 0 \qquad \forall k,$$

$$0 \leq (y_0^*A_{k2} + y_k^*B_{k2} + w_k^* - d_k) \perp z_k^{*\prime} \geq 0 \qquad \forall k,$$

$$0 \leq y_0^* \perp (\Sigma_k A_{k1}x_k^* + \Sigma_k A_{k2}z_k^{*\prime} - b_0) \leq 0$$

$$0 \leq y_k^* \perp (B_{k1}x_k^* + B_{k2}z_k^{*\prime} - b_k) \leq 0 \qquad \forall k,$$

$$w_k^*(z_k^{*\prime} - z_k^*) = 0 \qquad \forall k,$$

where "$0 \leq f(x) \perp x \geq 0$" is shorthand for the following complementarity condition for a scalar of column vector x and a function of the same dimension as x:

$$0 \leq f(x); \; x \geq 0; \; f(x)^T x = 0.$$

Now consider the following problem. Say that when the auctioneer defines **T**, each participant k is offered prices $\{y_0*, w_k*\}$ (term 2 of the contract), but their primal variables are unconstrained (term 1 is not enforced). Then each participant k will solve the following MIP that maximizes its benefits minus payment, subject to its internal constraints:

$$
\begin{aligned}
\textbf{PIP}_\mathbf{k} \quad & \text{Maximize: } v_{PIPk} = (c_k x_k + d_k z_k) - y_0*(A_{k1}x_k + A_{k2}z_k) - w_k*z_k \\
& \text{subject to:} \quad B_{k1}x_k + B_{k2}z_k \le b_k \qquad \forall k \\
& \qquad x_k \ge 0, \qquad \forall k \\
& \qquad z_k \in Z^k \qquad \forall k \quad .
\end{aligned}
$$

Let $v_{PIPk}*$ be the value of the objective of PIP $_k$ at $\{z_k*, x_k*\}$. We can show that $v_{PIPk}* = y_k*b_k$ as follows. Insert $\{z_k*, x_k*\}$ into the objective of PIP_k, and then add the term $y_k*(B_{k1}x_k* + B_{k2}z_k* - b_k)$ to the objective (which is permissible, since by the complementary slackness conditions given above, that term equals zero), and then cancel terms:

$$
\begin{aligned}
v_{PIPk}* &= (c_k x_k* + d_k z_k*) - y_0*(A_{k1}x_k* + A_{k2}z_k*) - w_k*z_k - y_k*(B_{k1}x_k* + B_{k2}z_k* - b_k) \\
&= (c_k - y_0*A_{k1} - y_k*B_{k1})x_k* + (d_k - y_0*A_{k2} - y_k*B_{k2}z_k* - w_k*)z_k* + y_k*b_k \\
&= y_k*b_k
\end{aligned}
$$

The third equality follows because the first and second terms in the second equality each equal zero by the complementary slackness conditions given earlier. Since both y_k and b_k are nonnegative, v_{PIPk} too is nonnegative, and all bidders must earn nonnegative (and perhaps positive) profits under contract **T**.

Now, let the optimal solution to PIP_k be $v_{PIPk}**$. If is $v_{PIPk}**$ is less than or equal to $v_{PIPk}*$ for each k, then the contract **T** is market clearing for the reasons below:

- no participant can obtain a feasible $\{x_k, z_k\}$ giving a greater profit in PIP_k than $\{x_k*, z_k*\}$, and

- as $\{x_k^*, z_k^*\}$ by definition solves PLIP, they also satisfy the market clearing condition $\Sigma_k(A_{k1}x_k + A_{k2}z_k) \le b_0$.

The last thing that must be shown is that $v_{PIPk}^{**} \le v_{PIPk}^*$ is indeed true. To demonstrate this, rearrange the terms of v_{PIPk}^{**} to yield the following:

$$v_{PIPk}^{**} = \text{Maximize } [(c_k - y_0^*A_{k1})x_k + (d_k - y_0^*A_{k2} - w_k^*)z_k]$$

$$\text{subject to:} \quad B_{k1}x_k + B_{k2}z_k \le b_k \quad \forall k$$

$$x_k \ge 0 \quad \forall k$$

$$z_k \in Z^k \quad \forall k .$$

Now let $\{x_k^{**}, z_k^{**}\}$ be the optimal solution for PIP_k. As a result, $v_{PIPk}^{**} = [(c_k - y_0^*A_{k1})x_k^{**} + (d_k - y_0^*A_{k2} - w_k^*)z_k^{**}]$. Now add the following nonnegative term to v_{PIPk}^{**}:

$$-y_k^*(B_{k1}x_k^{**} + B_{k2}z_k^{**} - b_k).$$

This term is nonnegative because $y_k^* \ge 0$ (see the PLIP complementary slackness conditions, above) and $B_{k1}x_k^{**} + B_{k2}z_k^{**} \le b_k$ (by the definition of PIP_k). As a result:

$$\begin{aligned} v_{PIPk}^{**} &\le [(c_k - y_0^*A_{k1})x_k^{**} + (d_k - y_0^*A_{k2} - w_k^*)z_k^{**}] - y_k^*(B_{k1}x_k^{**} + B_{k2}z_k^{**} - b_k) \\ &= [(c_k - y_0^*A_{k1} - y_k^*B_{k1})x_k^{**} + (d_k - y_0^*A_{k2} - y_k^*B_{k2} - w_k^*)z_k^{**}] + y_k^*b_k \\ &\le y_k^*b_k = v_{PIPk}^*. \end{aligned}$$

The last inequality results from noting that:

1. $(c_k - y_0^*A_{k1} - y_k^*B_{k1})x_k^{**} \le 0$, because $(c_k - y_0^*A_{k1} - y_k^*B_{k1}) \le 0$ (from the definition of DLIP, above) and $x_k^{**} \ge 0$.
2. $(d_k - y_0^*A_{k2} - y_k^*B_{k2} - w_k^*)z_k^{**} \le 0$, because $(d_k - y_0^*A_{k2} - y_k^*B_{k2} - w_k^*) \le 0$ (again from DLIP) and $z_k^{**} \ge 0$.

Consequently, we have shown that $v_{PIPk}^{**} \leq v_{PIPk}^{*}$; *i.e.*, no participant k can obtain a feasible solution giving a greater profit for PIP_k than the auctioneer's solution $\{x_k^*, z_k^*\}$.□

Theorem 2 shows that the dual solution to the constructed linear program, PLIP, can be used to form contracts for all bidders. In this auction, bidders may make or receive payments associated with their lumpy decisions. In contrast, in a traditional uniform price auction, a bidder is just paid the commodity (marginal) price (the dual variable on the market clearing constraint) for each unit of output and the dual variable on the individual capacity constraint is ignored.

Theorem 2 in the context of Scarf's problem provides other insights. For instance, there are some levels of quantity demanded for which both Smokestack and High Tech plants are "inframarginal" in the sense that their marginal costs are less than the commodity price and all are operating at capacity (see Table 3). In the linear program, the resulting scarcity rents appear as positive dual variables on binding upper bounds of activities. But, since they are paid an amount equal to what they bid, they are also marginal. In other words, if the technology were available to others and if any plant were to raise its bid, it would be replaced by another competitor.

In some of these instances, High Tech units can have negative start-up payments, indicating that scarcity rents from the commodity price exceed start-up costs. Such negative start-up payments can occur in order to dissuade uneconomic entry for plants on the margin. For instance, if any unit of a widely available type is collecting scarcity rents, then an infinite number of those units will wish to enter, and the market will not clear. However, in auctions where entry cannot occur instantaneously (e.g., daily power markets), then rents can be earned by units under a **T** contract even when, in the long run, the technology is widely available. In markets with integral constraints, the 'margin' may require the entire plant.

Examples of auctions formulated in a manner similar to, and yielding linear prices similar to **T** can be found in the New York Independent System Operator (NYISO) and the Pennsylvania-New Jersey-Maryland Interconnection (PJM) electric energy markets. In these markets, the market operator explicitly asks generators to bid costs associated with non-convexities (start-up and minimum load). In these markets, if a generating unit is started up in order to meet demand and if the revenues from the sale of energy fail to cover the sum of the variable costs and the startup costs, then the auctioneer provides a lump sum payment to the generator to make up the difference. On the other hand, if a generating unit's scarcity rents associated with binding internal capacity constraints are greater than start-up costs, then the generating units are allowed to keep the rents, effectively ignoring a negative dual price on the start-up constraint.

Theorem 3: If each participant k submits a bid reflecting its true valuations ($c_k x_k + d_k z_k$) and true constraints ($B_{k1} x_k + B_{k2} z_k \le b_k;\ x_k \ge 0;\ z_k \in Z^k$), an auction defined as follows maximizes net social benefits ($\Sigma_k\ [c_k x_k + d_k z_k]$) and is market clearing:

1. The auctioneer first solves problem PIP, yielding primal solution $\{x_k^*, z_k^*\}$;
2. The auctioneer determines prices $\{y_0^*,\ w_k^*\}$ by solving problem PLIP(z^*); and
3. The auctioneer offers contracts **T**.

Proof: By definition, the solution $\{x_k^*, z_k^*\}$ of PIP maximizes net social benefits and satisfies the second condition of market clearing ($\Sigma_k\ [A_{k1} x_k + A_{k2} z_k] \le b_0$). The only remaining condition is whether the prices from PLIP(z^*) support this solution. Theorem 2 demonstrates this for the payment schemes in **T**. ☐

Theorem 3 is an extension, to auctions with nonconvexities, of the Fundamental Theorem of Welfare Economics, which states that a competitive equilibrium is Pareto Optimal.

VI. Conclusions, Applications, and Extensions

This paper has addressed a problem that has troubled the economic analysis of markets with non-convexities: the existence of market clearing prices. Given the presence of non-convexities in emerging electricity auctions, this problem is of practical as well as theoretical interest. The contracts defined by **T** provide an answer to Scarf's (1994) search for a set of prices in the presence of non-convexities that yield zero profits for all activities in the optimal solution. These results hold for any market that can be represented by a mixed integer program.

Given recent advances in computational technology and integer programming algorithms, finding the prices necessary to define these contracts is practical. Roughly speaking, MIPs today take on average about the same or less time (wall clock) than linear programs of a similar size took to solve in the 1960s (Ceria, 2001; Hobbs, Stewart *et al.*, 2001). (With respect to computational times, the theoretical upper bounds on calculations have usually been much greater than the actual solution times for applications. There are several possible explanations for this discrepancy. First, it may be that actual applications seldom encounter pathological problems. Second, the difficult to solve problems are shelved. Third, difficult problems can often be reformulated to remove many pathologies.) Therefore, the results presented here are relevant for many practical problems. In particular, applying this approach to electric generating unit commitment auctions could be a significant step forward. As mentioned above, new and evolving electricity auction markets like PJM and NYISO have implemented market and pricing mechanisms similar to the one discussed in this paper.

Now that we can find prices that support an equilibrium in markets with non-convexities, there are many questions that can be examined. First, Scarf's (1994) search for price based tests for Pareto improving entry can be re-examined. If any potential activity can make a positive profit under the prices and quantities specified in contract **T,** then it should be included in the solution. Future work should investigate the definition and properties of such tests. However, such tests are unlikely to be both necessary and sufficient for evaluating the profitability of such activities in non-convex problems; in general, there may be some activities that fail those tests, yet their inclusion would still increase profit. A definitive test is to include the activity in the MIP and resolve the model. Fortunately, improved capabilities in mixed integer programming make that a more practical approach than it once was.

Second, much has been made in the electricity industry about the possibilities for strategic bidding behavior to manipulate prices (e.g., Borenstein and Bushnell, 1999). Adding bidding parameters, such as an integral activity like start-up costs, gives generators another degree of freedom that they can manipulate strategically. One intuitive observation can be made about strategic behavior. In the context of a sellers auction where the technologies are widely available and entry is instantaneous (as in the Scarf example in Section IV), even if the participants are not constrained to bid costs, a MIP auction solution produces a Nash equilibrium in which all generators bid their costs. The reason is that if anyone bids above its costs it would be immediately undercut by an entrant with the same costs. However, while this may be a good point of departure, the reality of market power in markets with integral activities is much different. An examination of whether a greater exercise of market power, and hence higher market power rents, are possible in the auction market proposed in this paper versus simple auctions in which non-convexities are ignored is required to address the above issue. In the context of such a study, issues like what bid parameters (integral or continuous) should be bid strategically to maximize profit, and what kind of activity rules hinder or help such strategic behavior. Moreover, the auction pricing mechanism proposed in this paper could be compared to first-price and Vickrey-Clarke-Groves auction mechanisms. See Hobbs *et al.* (2000) for a start at this.

Third, the efficiency of the auction pricing mechanism proposed here can be compared to the efficiency of simple auctions that ignore non-convexities. In particular, an efficiency comparison of the MIP based auction to a simple commodity price, one-time auction would be of interest. In the context of electricity markets, the above comparison may have interesting implications. While the overall cost impact of non-convex decisions may be small, these costs can be a significant portion of total costs to generators serving peak load or reliability functions. Moreover, without this mechanism, generators may receive physically infeasible dispatch orders.

Finally, our results say nothing about the uniqueness of equilibrium prices. In fact, as can be seen in Scarf's example in Section IV, there can be multiple equilibria. (In simple examples, degeneracy of the augmented LP can be a problem leading to multiple dual solutions. However, in larger more complex problems, it is not entirely clear how big a problem a multiplicity of solutions will be. In Scarf's example, the multiple equilibria result from the assumed identity of costs of different suppliers. In reality, costs and bids are seldom exactly equal.) Alternative equilibrium prices might lead to different distributions of surplus for market participants under contract **T**. Given that there is a lot of money at stake in the new electricity markets, where the bidding of non-convex costs is already taking place, an examination of the distributional consequences and efficiency of alternative equilibria and of suboptimal approaches is of keen interest to these market participants.

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